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OPTIMAL CONTROL OF DISTRIBUTED NUCLEAR REACTORS USING FUNCTIONAL ANALYSIS

BY

© ROLANDO NIEVA GÓMEZ

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled "Optimal Control of Distributed Nuclear Reactors Using Functional Analysis" submitted by Rolando Nieva Gómez in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

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ABSTRACT

The minimum norm formalism of functional analysis is applied to the problem of controlling the neutron flux distribution in nuclear reactors.

The application of the optimization technique is first described in terms of a general linear distributed parameter model. The general results thus obtained are later particularized to the specific case of a homogeneous slab reactor with the one-energy neutronic model.

The reduction of distributed parameter systems into decoupled and more tractable subsystems is discussed. It is demonstrated how symmetry principles can facilitate the practical implementation of the optimization techniques to more realistic distributed models which may include for example reactor configurations in two or three dimensions.

A new model expansion method is developed for solving the system of partial differential equations that describe the transient behavior of the most important fission product concentrations in thermal nuclear reactors.

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CHAPTER I

INTRODUCTION

The thermal-power distribution in the core of a large nuclear reactor depends on the fission-reaction rate at each point of the core and it is neither homogeneous nor stationary.

The rate at which fission reactions are produced is influenced by the neutron flux distribution. In turn this is affected by any change in the way neutrons are absorbed, diffused or produced throughout the core. These changes may be induced by a variety of causes. Power level adjustments, on-power refueling operations, fuel consumption and the accumulation of fission products in the nuclear fuel are typical examples.

In most reactor designs, the neutron flux distribution is controlled by introducing (or extracting) a neutron-absorbent substance at specific locations of the core.

The main objective for the reactor control system is to maintain the neutron-flux distribution within acceptable limits in order to avoid potentially harmful spots with high power density.

It is recognized that optimal control theory can play an important role in improving the performance of existing reactor control systems and considerable research has been done in this area.

The work presented in this thesis is a contribution to that research effort.

1 Scope of the thesis

Chapter II is a review of the abstract minimum norm problems of functional analysis. The solution to particular cases which are utilized in later chapters are briefly discussed.

Chapter III is devoted to the application of the minimum norm formalism to the problem of bringing the neutron flux distribution from an initial state into the vicinity of a desired distribution while minimizing a performance index that penalizes both the deviations of the flux from the desired distribution and the control effort. This problem is relevant to the particular situation where an external disturbance, such as on-power refueling, has affected the flux distribution and it is desired to reshape the resulting pattern into its original shape. This problem has been discussed in the past by several authors who utilized a variety of different optimization techniques. A review of their work is given in the chapter with the purpose of providing a yard-stick against which the optimization approach presented in this thesis can be compared.

The application of the minimum norm approach is made first in terms of a very general linear distributed parameter model. The application is later particularized to the specific case of a homogeneous slab reactor with the one-energy neutronic model and a finite number of control rods.

In Chapter IV several variations on the problem of controlling the neutron-flux are discussed. In addition to the requirements of the problem treated in Chapter III, the problem formulations in Chapter IV include new constraints imposed on both the total power generated in

the core and the state of the reactor. These constraints relate to the particular situation where it is desired to change or adjust the total power output from the core in a finite interval of time while minimizing both the distortion of the flux distribution and the control effort. It is demonstrated in this chapter how the additional constraints can be treated in the context of the minimum norm formalism.

The application of the optimization technique is described in terms of a general linear distributed parameter model. Several examples are also given, in which the particular case of a homogeneous slab reactor with the one-energy neutronic model is considered.

Chapters V and VI are devoted to the development of computational methods that will facilitate the practical implementation of the optimization techniques to more realistic reactor models, which may involve for example geometrical configurations in two or three dimensions.

In Chapter V the reduction of mathematical models into decoupled and more tractable subsystems is discussed. The reduction procedure employed there relies on group theoretic arguments and symmetry principles.

A short summary of the group theoretic results that are needed to describe the reduction approach is given for reference purposes.

It is demonstrated in this chapter how by merely studying the geometrical symmetry of a nuclear reactor core it is possible to decouple the conditions for optimality that result from the application of the minimum norm technique.

The reduction of a cylindrical reactor with the one-energy neutronic model is also given as an example.

A method is developed in Chapter VI for obtaining an approximate solution to a reactor core model that is suitable for describing the dynamic behavior of the reactor core during load-following operations.

The method, which can be classified as a modal expansion approach, uses the eigenfunctions of the Laplacian operator and yields the solution to the model in an operator form that is amenable to application of the minimum norm technique.

Both the symmetry reduction procedure of Chapter V and the modal expansion approach are combined in an example that involves a cylindrical reactor and utilizes the numerical data typical of a large, pressurised-water, natural-uranium nuclear reactor.

CHAPTER II

THE ABSTRACT MINIMUM NORM PROBLEMS OF FUNCTIONAL ANALYSIS

1. Introduction

In contrast with other optimization techniques, which generally approach problems of a particular dynamic nature, functional analysis with its geometric character provides a unified framework for optimization problems of discrete, continuous, distributed or composite nature.

This chapter constitutes a review of a class of functional analytical problems which form part of the well-developed optimization theory by vector space methods, the so-called abstract minimum norm problems.

Although many optimization problems cannot be formulated in these terms, the availability of a large variety of different norms provides enough flexibility for the minimum norm formulation to be of importance, particularly in the field of optimal control.

For reference purposes, the solution to certain particular cases which shall prove to be useful in later chapters, will be discussed in the next section.

2. The Class of Abstract Minimum Norm Problems

Historically the method of moments may be seen to be the first functional analytical technique, used in connection with optimal control problems, in which the minimum norm formulation is employed. The origin of the method of moments can be traced back to the very early developments in functional analysis.

For reference purposes the problem of moments is stated below.

Problem 1. The ℓ-problem of moments

Given the linearly independent elements f_i, $i=1,2,\ldots,N$ in a normed linear space B, the real numbers α_i, $i=1,2,\ldots,N$ and the positive real number ℓ find the necessary and sufficient conditions for there to exist a linear functional u defined on B satisfying the constraint

$$u(f_i) = \alpha_i \qquad i=1,2,\ldots,N \tag{1}$$

and with norm

$$||u|| = \sup_h \frac{|u(h)|}{||h||} \leq \ell \tag{2}$$

In his book *Theorie des operations Linéaires* (reference [48], pp. 57, 74-75), Banach discussed this problem and mentioned the work of Riesz [63] and Helly [64] concerning the necessary and sufficient conditions for there to exist a functional satisfying the conditions of problem 1.

Krein [49] studied the problem in a very general manner and gave specific results for certain normed spaces. It is in his honour that

problem 1 is often referred to in this literature as "Krein's ℓ-problem of moments".

The first application of Krein's results to optimum control theory were made by Krasovskii in connection with the time optimal control of a linear plant with amplitude-constrained control signals [51].

Krasovskii's ideas triggered the publication of a large number of papers which were concerned with amplitude, energy, power and fuel constraints on both the control signals and the system states as well. A modern exposition of the method of moments and/or a literature review on its application to optimum control theory can be found in the references [45, 53, 59].

For comparison purposes, the ℓ-problem of moments may be formulated in a different manner;

Problem 1(a).

Let x be a given vector of the n-dimensional euclidean space E_n, let c be a convex set of linear functionals defined on the normed space B,

$$c = \{u : ||u|| = \sup_{h} \frac{|u(h)|}{||h||} \leq \ell,\ h\epsilon B\} \tag{3}$$

then for a given linear bounded transformation F mapping c into E_n, find the necessary and sufficient conditions for there to exist a $u\epsilon c$ satisfying

$$Fu=x \tag{4}$$

Motivated by the class of linear final value problems in optimal control, Balakrishnan discussed in his classical paper [52] an abstract minimum norm problem of a more general scope,

Problem 2.

Let F be a compact linear bounded transformation mapping a Hilbert space H_1 into another Hilbert space H_2. For a given x in H_2 it is required to minimize

$$||Fu - x||^2 \tag{5}$$

subject to u being in the sphere c in H_1 defined by

$$||u||^2 \leq \ell^2 \tag{6}$$

This formulation may be seen to be a variation on the ℓ-problem of moments in the sense that the control u is not required to be a solution to the equation

$$Fu=x \tag{7}$$

but only the best approximation in the H_2-norm.

The compactness assumption on F gave this problem a very general scope since the range of F was no longer required to lie in a finite dimensional space. The solution to this problem was obtained by Balakrishnan [52] in the form given below.

Solution to Problem 2.

Either

$$\sup_{k>o} ||[F^*F+kI]^{-1} F^*x|| \leq \ell \tag{8}$$

in which case the sequence

$$u_k = [F^*F+kI]^{-1}F^*x \tag{9}$$

is such that u_k *converges to the optimal element* u_o *of minimal norm*

$$\lim_{k\to o} ||Fu_k-x||^2 = \inf_{u\in c} ||Fu-x||^2 = ||Fu_o-x|| , \tag{10}$$

or

$$\sup_{k>o} ||[F^*F+kI]^{-1} F^*x|| > \ell \tag{11}$$

in which case

$$u_o = [F^*F+k_oI]^{-1} F^*x \tag{12}$$

where k_o *is adjusted so that* $||u_o||=\ell$ *yields the unique solution to problem 2.*

In a second paper Balakrishnan [58] extended these results to a Banach space setting and, using the theory of one-parameter semigroups, developed a general theory of optimum control for mathematical models described by the abstract Cauchy problem. Also in this paper, the compactness assumption on F was removed and, by invoking the properties

of convex functionals in Hilbert spaces, the solution to problem 2, given above, was shown to remain valid.

Motivated by the basic mathematical structure of linear systems, Porter [60] presented a new approach to the general minimum energy problem in Hilbert spaces. Porter and Williams [55] extended the results of the previous paper to a Banach space setting and discussed the questions of existence, uniqueness and the properties of the optimal control. The problem treated by Porter and Williams is stated as follows;

Problem 3.

Let B and D be Banach spaces and T, a bounded linear transformation defined on B with values in D. For each ξ in the range of T find an element $u \in B$ that satisfies

$$\xi = Tu \tag{13}$$

while minimizing the performance index

$$||u|| \tag{14}$$

This formulation may be seen to be a generalization of the ℓ-problem of moments in that the range of T is not required to lie in a finite dimensional space but in a general Banach space. The control u is required however, as in the problem of moments, to satisfy a given equation.

In the earlier developments of Porter's work the assumption that T is a linear bounded and onto transformation is always present.

Knowing that distributed parameter systems have mathematical models which typically involve a transformation with dense range, Porter [57] removed the assumption of T being onto and considered the case of a linear bounded transformation with dense range. It is proved in his paper that the results previously obtained are still valid under the new assumption.

Several variations on the general minimum energy problem, which dramatically enlarged the application scope of the abstract minimum normed formulations, were considered by Porter and Williams [56]. The most general of these variations is, for reference purposes given below;

Problem 4.

Let F be a bounded linear transformation from the Banach space B into the Banach space B_1, let T be a bounded linear transformation from B onto (or with dense range) the Banach space D, and let $\hat{u}$, $\hat{y}$ and ξ be given vectors in B, B_1 and D respectively. Find a u in B satisfying

$$Tu=\xi \tag{15}$$

that minimizes

$$||u-\hat{u}||^2+||Fu-\hat{y}||^2 \tag{16}$$

The questions of existence and uniqueness as well as the properties of the optimal control are discussed in references [56] and [32].

Of particular interest to the application considered in this thesis is the Hilbert space version of problem 4, in which case the unique

solution obtained by Porter and Williams (see reference [32], pp. 348) is as follows;

Solution to Problem 4.

The unique solution u_o of the Hilbert space version of problem 4 is given by

$$u_o=[I+F^*F]^{-1}\ (T^+\eta+\hat{u}+F^*\hat{y}) \tag{17}$$

where η is the unique vector in D satisfying

$$\xi=T[I+F^*F]^{-1}\ (T^+\eta+\hat{u}+F^*\hat{y}) \tag{18}$$

and where F^* is the adjoint of F. T^+ is the pseudo inverse of T defined by

$$T^+\xi=T^*[TT^*]^{-1}\xi \tag{19}$$

Provided that TT^* is invertible.

CHAPTER III

OPTIMAL CONTROL OF DISTRIBUTED NUCLEAR REACTORS

1. Introduction

This chapter is devoted to the problem of controlling the neutron flux distribution in a nuclear reactor core where the spatial kinetic effects are important.

The optimization technique of the minimum norm problem in Hilbert spaces is applied to the problem of adjusting the neutron flux for a general distributed nuclear reactor whose dynamic behavior is described in the neighborhood of an equilibrium condition.

The problem consists of computing the control function that ttansfers the state of the system from an initial condition to a desired state, in a given period of time, and minimizes a quadratic performance index that penalizes the deviations from equilibrium so as to avoid potential spatial instabilities and high power density spots.

2. Background

Kliger [65] in 1965 considered the problem of changing the neutron density level in such a way as to keep its spatial shape undistorted. The performance index to be minimized was the mean-square error in space and time, between the actual system response and the desired trajectory. The reactor core was modelled by the one-energy neutron group model with delayed neutron precursors. The model's parameters were assumed to be independent of the spatial coordinate,and a pseudocontrol function proportional to the product between the neutron multiplication factor and the neutron density was treated like a distributed control. Both the system state and the control were expanded in terms of an orthonormal set of spatial modes. With the spatial independence of the system parameters, a decoupled set of ordinary differential equations was obtained for the expansion coefficients. The control term for each one of the spatial modes was determined by making use of known results for point reactor models. The distributed pseudocontrol effect thus determined was later approximated by a finite number of spatially concentrated control rods. In spite of the simplicity of the model considered, Kliger's contribution holds the merit of having introduced several features which have been used by many authors, and in a sense became standard techniques, namely, the use of an orthonormal expansion and the treatment of a pseudocontrol function like a distributed input.

In 1966 Wiberg [74,75,11] studied the optimal feedback control of spatial xenon oscillations. Wiberg considered a very general but linear reactor model and made use of a modal expansion in terms of the Kaplan modes [7]. The finality property that characterizes this set of modes allowed for the decoupling of the power control system from the spatial regulator. The spatially concentrated control functions were modelled by

means of a first order perturbation. Upon truncation to a finite number of modes and making use of a quadratic performance index, which penalized the state deviations from equilibrium and measured the control energy, the control problem was reduced to terms which are manageable by the theory of optimal control by state space methods [76]. Wiberg's major contribution is the idea of decoupling the power control system from the spatial regulator-system. His chapter in [11] has become one of the most important references in the field. Hsu and Bailey [66] in 1967 applied an extended version of the maximum principle to a simple one-group model for an homogeneous slab reactor with one delayed neutron precursor model. A quadratic cost functional was considered. Their approach is based on Wang's work on distributed parameter systems [46].

Stacey applied several optimization techniques to the problem of controlling xenon spatial transients in thermal reactor cores. In 1968 [76], he treated the problem in terms of the dynamic programming formalism and gave a numerical example for a fairly realistic 3-dimensional reactor model. The large number of state variables present, however, made it necessary to severely limit the set of allowable controls.

In 1969, Stacey [77] treated the xenon oscillation problem by using the calculus of variations for distributed parameter systems. In this case the optimality conditions were obtained in the form of a system of partial differential equations with mixed boundary conditions, which are very difficult to solve. A numerical example was given for the one-group model with xenon and iodine dynamics. A distributed control input and a quadratic cost functional were considered in the example, for which an approximate solution was obtained by applying the methods

of quasilinearization and nodal approximation. Knowing the large difference in the time constants of the neutron kinetics and the reactor poisoning process, Stacey assumed in his work that changes in the neutron flux and temperature distributions occur instantaneously. In 1970, Stacey [78] considered the application of variational synthesis to the optimal control of spatially dependent reactor models. The so-called method of variational synthesis consists of expanding the system state and the control input in terms of known functions of space and time. The conditions that would minimize a given performance index are obtained by simple calculus, in the form of algebraic equations. The choice of expansion functions in this case is arbitrary and the approach lacks the rigor of other methods.

Chauduri [79] in 1972 applied the maximum principle to the problem of controlling xenon transients. A quadratic cost functional was considered and a distributed control input was assumed. Solutions were obtained for a one-dimensional reactor model via space and space-time discretization.

Lazarevich [80] et. al. in 1972 considered the application of dynamic programming methods to the problem of controlling the flux distribution. Results were obtained for a slab reactor by applying Galerkin's method to the Hamilton-Jacobi canonical equations. An algorithm for computing the optimal control using the eigenvectors of the Hamilton-Jacobi operator was also proposed. The approach can be seen to be equivalent to Kaplan's modal expansion method [7].

In 1974, El-Bassioni and Poncelet [13] treated the minimum time control of spatial xenon oscillations via Pontryagin's minimum principle

and Kaplan's modal expansion method. The same problem was considered by Chiang et. al. in 1976 [15] but this time the lambda-mode analysis was applied in order to circumvent the problem of computing the Kaplan modes.

In 1973 Iwazumi and Koga [18] considered the terminal control problem for a slab reactor with the one-group diffusion and one delayed neutron precursor model. The idea of introducing a pseudocontrol function was suggested, so as to replace the spatially dependent parameters of the model by both a combination of spatially independent parameters and a particular initial pseudocontrol distribution. The modal expansion method in terms of the "clean" reactor modes was applied and the familiar Riccati equations were derived for each mode. Motivated by the relatively large difference in the time constants of the prompt and delayed neutron kinetics, recently (1977), Assatani et. al. [19] suggested in connection with Iwazumi's approach, a singular perturbation method for solving the Riccati equations [82,83].

The first application of the abstract minimum norm formalism to the problem of controlling the neutron flux distribution was made in 1968 by Kyong [67]. Kyong treated the terminal control problem for a reactor core of cylindrical configuration, containing a finite number of control rods and with the neutron kinetics modelled by the one-group diffusion equation. Although the constraint on the maximum allowable control effort was not explicitly stated, Kyong's formulation of the problem may be seen to correspond to the abstract minimum norm problem treated by Balakrishnan in [52] and referred to as problem 2 in Chapter II of this thesis. The unique optimal control, for the terminal problem, was shown

to satisfy a coupled set of Fredholm's integral equations of the second kind. A rigorous method, based on the characteristic expansion [68], was proposed for solving the integral equations. It is clear from Kyong's work that the functional analytical formulation yields necessary and sufficient conditions for optimality in a form which is amenable to application of a different class of computational techniques, which in some cases may prove to be superior to the more conventional computational methods associated with the variational formulation.

In 1973, Iwazumi and Koga also discussed the application of the minimum norm formulation to the terminal control problem, but this time a slab reactor model, with distributed control input and the one group - one delayed precursor neutron kinetics was considered.

The relevance of the terminal control problem to reactor core control applications is questionable. Given that the neutron flux deviations from a desired state are only penalized at the end of the control interval, it is possible that large power transients could result from the implementation of the control function which is optimum for the terminal problem. This observation motivated the author to study the application of the minimum norm formulation to the case where the neutron flux distribution is brought from an initial state to a desired equilibrium distribution while penalizing both the deviations from equilibrium along the trajectory and the control effort. Some results of this investigation were published in 1977 by the author, Christensen and El-Hawary in reference [17]. This chapter and the next constitute an expanded version of that work.

3. The Reactor Core Model

Wiberg [11] described the mathematical model of a nuclear reactor core and discussed the nature of the most important kinetic processes.

For convenience, the linearized model described by Wiberg will be considered here in its most general form. In order to avoid unnecessary repetition, the reader is referred to [11] for the details of the derivation. Also for reference purposes, the specialization of the model to the case of two neutron groups with delayed neutron precursors and xenon and iodine dynamics is considered in [7] and [11].

The dynamical equation of a general reactor near an equilibrium state is given by a linear vector partial differential equation of the form

$$\frac{\partial}{\partial t}\psi(r, t) = A(r)\,\psi(r, t) + B(r)\,U(t) \qquad (1)$$

where r is in general a 3-dimensional spatial variable, defined on the reactor extrapolated volume. $\psi(r, t)$, for fixed r and t is an N-dimensional vector containing; the neutron flux at different energies, delayed neutron precursor densities, fission product and precursor densities, and fuel and moderator temperatures. A(r) is a matrix spatial operator involving the gradient and Laplacian operators. For fixed time t, U(t) is an M-dimensional vector representing the effect induced by the control rods. B(r) is a rectangular matrix spatial operator of the appropriate dimensions.

Associated with equation (1) are boundary conditions of the form

$$\psi(r, t) = 0 \qquad (2)$$

at the reactor extrapolated boundary, and the initial condition

$$\psi(r, t_0) = Z_0(r) \qquad (3)$$

at time t_0. The neutron flux and current must also satisfy continuity conditions at internal boundaries.

In general, one wishes to find a solution to equation (1) in a normed function space of interest. Here the particular case is considered where for each fixed time t, the state $\psi(r, t)$ is a function in a real Hilbert space H, endowed with an inner product of the form

$$< Z(t), W(t) >_H = \int_V W^T(r, t)\, Q(r)\, Z(r, t)\, dr \qquad (4)$$

in which the integral is over the core extrapolated volume V. $Q(r)$ is an N-dimensional, positive definite matrix with space-dependent entries. W^T denotes the transpose of W.

Similarly the control function U(t), for a fixed time t, is assumed to be an element of a finite dimensional, real Hilbert space E, with inner product

$$< U(t), Y(t) >_E = Y^T(t)\, R\, U(t) \qquad (5)$$

where Y^T is the transpose of Y, and R is a positive definite matrix of the appropriate dimension. The control functions considered belong to the class of functions with a bounded norm of the form

$$\int_{t_0}^{t_1} U^T(t)\, R\, U(t)dt = \int_{t_0}^{t_1} ||U(t)||_E dt \qquad (6)$$

The assumption is made that equation (1), together with the associated boundary conditions, constitute a well posed abstract Cauchy problem in relation to the norm topologies of H and E. Therefore the unique solution to (1) and (2) can be written in the form

$$\psi(r, t) = G(r, t; r', t_0)\, Z_0(r') + \int_{t_0}^{t} G(r, t; r', \tau) B(r')\, U(\sigma)d\tau \qquad (7)$$

where $G(r, t; r', \tau)$ is the strongly continuous semigroup associated with the operator $A(r)$, which satisfies,

$$G(r, t_1; r', \tau)\, G(r, t_2; r', \tau) = G(r, t_1+t_2; r', \tau) \qquad (8)$$

$$G(r, t; r', t) = \text{Identity} \qquad (9)$$

and is also a strong solution to

$$\frac{\partial}{\partial t} G(r, t; r', \tau)\, \psi(r', \tau) = G(r, t; r', \tau)\, A(r')\, \psi(r', \tau) \qquad (10)$$

in the Hilbert space H.

Equation (7) is the solution to (1) in the sense that ψ satisfies the relations

$$\lim_{h \to 0_+} \left|\left| h^{-1}[\psi(t+h) - \psi(t)] - A\,\psi(t) - B\,U(t) \right|\right|_H = 0 \qquad (11)$$

for all $t \geq t_0$, and where $h \to 0_+$ denotes $h \to 0$ through positive values, and

$$\lim_{t \to 0_+} \left|\left| \psi(t) - Z_0 \right|\right|_H = 0 \qquad (12)$$

for all initial conditions Z_0 in the domain of A;

$$D[A(r)] = \{Z(r) \; ; \; \left|\left| A(r)\, Z(r) \right|\right|_H < \infty, \text{ and } Z(r) = 0$$

in the boundary of v}.

Several remarks are pertinent;

1. - Balakrishnan [58] and Wang [41] discuss the well-posedness question in connection with the abstract Cauchy problem in Banach spaces, and give sufficient conditions for the unique solution to be in the form of equation (7). Namely it is required that: a) A(r) is the infinitesimal generator of the semigroup G, b) BU is strongly measurable and Bochner integrable in every finite time interval, c) BU is in D[A] for almost every t and d) ||ABU|| is integrable in each finite time interval.

A detailed discussion on these definitions and properties can be found in reference [89].

2. - When the matrices Q and R, appearing in the inner product definitions (5) and (4), are chosen to be the identity in their corresponding spaces, it is clear that the solution (7) would satisfy (1) in the

mean square sense and the control functions being considered would belong to the class of controls with limited energy. The inclusion of a matrix Q other than the identity, provides a wider range of norms, which would allow for example, to emphasize the difference between the state distributions and the equilibrium condition in a given region of the reactor where core-damage due to large flux deviations is more likely to occur.

3. - Although it is convenient to describe the application of optimization techniques to reactor control problems in terms of a general model, that at least in theory applies to all possible situations, it should be pointed out that a model of such generality is seldom used in reactor simulations, for the simple reason that it is too complex to treat computationaly. In practical situations, approximations of diverse nature are always introduced. Of these, the most common is the hierarchical decomposition of the general model according to the time scale that is relevant to the particular situation or problem in case.

4. - If the conditions of remark 1 are satisfied, then the solution to the mathematical model exists and it can always be obtained in the form of an integral expression of convolution type such as (7). However, it should be pointed out that the conditions of remark 1 are difficult to verify in practical situations. Also, obtaining the solution in the form of (7) is seldom an easy task.

5. - Finally a few explanatory remarks on notation are mandatory. The semigroup $G(r, t; r', \tau)$ represents in our case an integral operator. The kernel of this operator depends on the four variables r, t, r' and τ. Of these, only r' defines the dummy variable of integration.

4. The Optimal Control Problem

This section is devoted to the control problem where the state distribution is brought from an initial state to a desired equilibrium condition, while penalizing both the deviations from equilibrium along the trajectory and the required control effort.

The performance index considered here is given by

$$J(U)= \int_{t_0}^{t_1} \int_V [\psi(r, t) - Z(r)]^T Q(r)[\psi(r, t) - Z(r)]drdt$$

$$+ k_0 \int_{t_0}^{t_1} U^T(t) R U(t)dt \qquad (14)$$

where Z is the desired state. k_0 is a positive constant chosen so as to establish the relative weight of the second term in (14). ψ is the system's state given by equation (7). Q and R are defined as before.

It is convenient to introduce the following notation: H_1 and H_2 will denote the Hilbert spaces with inner products

$$\langle U, Y\rangle_{H_1} = \int_{t_0}^{t_1} \langle U(t), Y(t)\rangle_E dt \qquad (15)$$

and

$$\langle Z, W\rangle_{H_2} = \int_{t_0}^{t_1} \langle Z(t), W(t)\rangle_H dt \qquad (16)$$

respectively. Also, the operator F, mapping H_1 into H_2 will be defined by

$$F(r, t; \tau)\, U(\tau) = \int_{t_0}^{t} G(r, t; r', \tau)\, B(r')\, U(\tau) d\tau \qquad (17)$$

With this notation, the cost functional given in (14) can be rewritten in the form

$$J(U) = ||FU - Z + G_0 Z_0||_{H_2} + k_0 ||U||_{H_1} \qquad (18)$$

where Z_0 is the state distribution at time t_0 and G_0 is the semigroup defined by

$$G_0 = G(r, t; r', t_0) \qquad (19)$$

which maps the space H_2 into itself.

The minimum norm formulation of the control problem can be seen to correspond with problem 2 of the previous chapter. Therefore the optimal control U can be determined from

$$[F^*F + k_0 I]U = F^*[Z - G_0 Z_0] \qquad (20)$$

where F* is the operator adjoint of F.

4.1 The Adjoint F*

The adjoint F* is related to F through the inner product relation

$$\langle FU, Z \rangle_{H_2} = \langle U, F^*Z \rangle_{H_1} \qquad (21)$$

the left side of (21) can be expanded as follows

$$\langle FU, Z\rangle_{H_2} =$$

$$\int_{t_0}^{t_1} \langle \int_{t_0}^{t} G(t;\tau)\, B\, U(\tau)\, d\tau,\ Z(t)\rangle_H dt \qquad (22)$$

where for simplicity only the temporal dependency is explicitly shown. The inner product in H involves an integral operation throughout the reactor core volume. After interchanging the order of integration, one finds that (22) can be written in the form

$$\langle FU, Z\rangle_{H_2} =$$

$$\int_{t_0}^{t_1} \int_{t_0}^{t} \langle G(t;\tau)\, BU(\tau),\ Z(t)\rangle_H d\tau dt \qquad (23)$$

which in terms of the adjoint operators G* and B* becomes

$$\langle FU, Z\rangle_{H_2} = \int_{t_0}^{t_1} \int_{t_0}^{t} \langle U(\tau),\ B^*G^*(t;\tau)\, Z(t)\rangle_E d\tau dt \qquad (24)$$

and after interchaning the order of integration

$$\langle FU, Z\rangle_{H_2} = \int_{t_0}^{t_1} \int_{\tau}^{t_1} \langle U(\tau),\ B^*G^*(t;\tau)\, Z(t)\rangle_E dt d\tau \qquad (25)$$

$$= \int_{t_0}^{t_1} \langle U(\tau),\ \int_{\tau}^{t_1} B^*G^*(t;\tau)\, Z(t) dt\rangle_E d\tau \qquad (26)$$

from which F* is obtained as follows,

$$F^*(\tau, t)Z(t) = \int_{\tau}^{t_1} B^*G^*(t; \tau)\, Z(t)dt \tag{27}$$

Similarly, B* is related to B through the relation

$$\langle BU, Z\rangle_H = \langle U, B^*Z\rangle_E \tag{28}$$

Expanding the left side of (28) one finds that,

$$\langle BU, Z\rangle_H = \int_V Z^T(r)\, Q(r)\, B(r)Udv \tag{29}$$

$$= \int_V [B^T(r)\, Q^T(r)\, Z(r)]^T dv\, U \tag{30}$$

$$= [R^{-1} \int_V B^T(r)\, Q^T(r)\, Z(r)dv]^T RU \tag{31}$$

$$= \langle U, R^{-1} \int_V B^T(r)\, Q^T(r)\, Z(r)dv\rangle_E \tag{32}$$

from which it follows that the adjoint B^* is given by

$$B^* = R^{-1} \int_V B^T(r)\, Q^T(r)\, Z(r)dv \tag{33}$$

where R has been assumed symmetric.

4.2 The Necessary and Sufficient Condition for Optimality: A Fredholm's Integral Equation

Substituting F and F*, as obtained in expressions(17) and (27), one finds that the term F^*FU can be expanded in the form

$$F^*FU = \int_{\tau}^{t_1} B^*G(t;\tau) \int_{t_0}^{t} G(t;\alpha)\, BU(\alpha)\,d\alpha dt \qquad (34)$$

where again for simplicity only the temporal dependency is explicitly shown.

Interchanging the order of integration in the expression (34), one obtains

$$F^*FU = \int_{t_0}^{\tau} \int_{\tau}^{t_1} B^*G^*(t;\tau)\, G(t;\alpha) B dt\, U(\alpha) d\alpha$$

$$+ \int_{\tau}^{t_1} \int_{\alpha}^{t_1} B^*G^*(t;\tau)\, G(t;\alpha)\, Bdt\, U(\alpha) d\alpha \qquad (35)$$

which, in turn, can be rewritten in the form

$$F^*FU = \int_{t_0}^{t_1} K(\tau;\alpha)\, U(\alpha) d\alpha \qquad (36)$$

where

$$K(\tau,\alpha) = \begin{cases} \int_{\tau}^{t_1} B^*G^*(t,\tau)\, G(t;\alpha) B dt \text{ for } \alpha<\tau \\ \\ \int_{\alpha}^{t_1} B^*G^*(t;\tau)\, G(t;\alpha) B dt \text{ for } \alpha \geq \tau \end{cases} \qquad (37)$$

It follows from expressions (20) and (36) that the condition for optimality is given by a Fredholm's integral equation of the second kind,

$$U(\tau) = \Delta(\tau) - \frac{1}{k_0} \int_{t_0}^{t_1} K(\tau;\ \alpha)\ U(\alpha)d\alpha \qquad (38)$$

where the function Δ is defined by

$$\Delta(\tau) = \frac{1}{k_0} F^*[Z - G_0 Z_0]$$

$$= \frac{1}{k_0} \int_{\tau}^{t_1} B^* G^*(t;\ \tau)\ [Z - G_0 Z_0]dt \qquad (39)$$

It can be noticed from the defining relation (37) that $K(\tau, \alpha)$ is not in the form of a product or a sum of products of functions that depend on a single variable. In other words, $K(\tau, \alpha)$ is a non-degenerate kernel.

It is worth mentioning here that the necessary and sufficient conditions for optimality derived by Kyong [67] for the terminal control problem are in the form of an integral equation with a degenerate kernel.

Since the key ingredient in the characteristic expansion method proposed by Kyong for solving the optimality conditions is precisely

the degeneracy of the kernel, it follows that this approach is not applicable to equation (38).

It should be pointed out however, that equation (38) is amenable to application of approximating techniques such as contraction mapping algorithms [90], or methods that reduce the integral equation into a system of algebraic equations via temporal discretization or function expansion techniques [27].

4.3 An Example

Consider an homogeneous slab reactor model and the one-neutron group diffusion equation

$$\frac{1}{V}\frac{\partial \phi}{\partial t}(r, t) = D\frac{\partial^2}{\partial r^2}\phi(r, t) + [v\Sigma_f - \Sigma_a]\phi(r, t) - \sum_{i=1}^{M}\phi(r, t)u_i(t)\delta(r-r_i) \tag{40}$$

with boundary condition

$$\phi(0, t) = \phi(b, t) = 0$$

and the initial neutron flux distribution $\phi(r, t_0)$.

D is the neutron diffusion coefficient, Σ_f and Σ_a denote the fission and absorption macroscopic cross sections respectively. V is the average neutron velocity, v denotes the number of neutrons generated per nuclear fission. b denotes the reactor width. The control absorption cross section at the location r_i is represented by $u_i(t)$.

The performance index

$$J(U) = \int_{t_0}^{t_1}\int_{0}^{b}[\phi(r, t) - \phi_0(r)]^2 dr dt + k_0\int_{t_0}^{t_1}\sum_{i=1}^{M}u_i^2(t)dt \tag{41}$$

penalizes the flux deviations from the equilibrium distribution $\phi_0(r)$

$$\phi_0(r) = \phi_M[\sqrt{\frac{2}{b}} \sin(\frac{\pi r}{b})] \ , \tag{42}$$

where ϕ_M is a constant. Equation (42) is a solution to

$$0 = D \frac{\partial^2}{\partial r^2} \phi(r) + [v\Sigma_f - \Sigma_a]\phi(r) \tag{43}$$

subject to the boundary condition.

Small flux deviations from $\phi_0(r)$

$$\psi(r, t) = \phi(r, t) - \phi_0(r) \tag{44}$$

satisfy the linearized equation

$$\frac{\partial \psi}{\partial t}(r, t) = VD \frac{\partial^2}{\partial r^2} \psi(r, t) + V[v\Sigma_f - \Sigma_a]\psi(r, t) - V \sum_{i=1}^{M} u_i(t)\phi_0(r)\delta(r-r_i) \tag{45}$$

with the initial condition

$$\psi(r, t_0) = Z_0(r) = \phi(r, t) - \phi_0(r) \tag{45a}$$

and boundary conditions

$$\psi(0, t) = \psi(b, t) = 0 \tag{45b}$$

which can be solved by applying the method of separation of variables. The solution is given by

$$\psi(r, t) = \sum_n \frac{2}{b} \sin\left(\frac{n\pi r}{b}\right)\left[\int_0^b \sin\left(\frac{n\pi r'}{b}\right) Z_0(r')dr'\right]e^{\lambda_n(t-t_0)}$$

$$+ \left[\frac{2}{b}\right]^{3/2} V\phi_M \int_{t_0}^{t} \sum_n e^{\lambda_n(t-\tau)} \sin\left(\frac{n\pi r}{b}\right)\left[\sum_{i=1}^{M} \sin\left(\frac{n\pi r_i}{b}\right)\sin\left(\frac{\pi r_i}{b}\right)u_i(\tau)\right]d\tau \tag{46}$$

where

$$\lambda_n = [v\Sigma_f - \Sigma_a - \left(\frac{n\pi}{b}\right)^2 D]V \tag{47}$$

In terms of the notation introduced at the beginning of this section, the solution (46) becomes

$$\psi(r, t) = G(r, t; r', t_0)Z_0(r')$$

$$+ \int_{t_0}^{t} G(r, t; r', \tau)B(r')U(\tau)d\tau \tag{48}$$

where

$$G(r, t; r', t_0)Z_0(r') = \sum_n \frac{2}{b} \sin\left(\frac{n\pi r}{b}\right)\left[\int_0^b \sin\left(\frac{n\pi r'}{b}\right)Z_0(r')dr'\right]e^{\lambda_n(t-t_0)} \tag{49}$$

and

$$F(r, t; \tau)\, U(\tau) =$$

$$[\tfrac{2}{b}]^{3/2} V\phi_M \int_{t_0}^{t} \sum_n e^{\lambda_n (t-\tau)} \sin(\tfrac{n\pi r}{b}) [\sum_{i=1}^{M} \sin(\tfrac{n\pi r_i}{b}) \sin(\tfrac{\pi r_i}{b}) u_i(\tau)]\, d\tau \tag{50}$$

The operator $G(r, t; r', t_0)$, in this case, is self adjoint. The adjoint F* is given by an M-dimensional vector operator with entries $f_i^*(\tau; r, t)$ defined by

$$f_i^*(\tau; r, t) Z(r, t) =$$

$$V\phi_M (\tfrac{2}{b})^{3/2} \int_{\tau}^{t_1} \int_{0}^{b} \sum_n e^{\lambda_n (t-\tau)} [\sin(\tfrac{n\pi r_i}{b}) \sin(\tfrac{\pi r_i}{b})] \sin(\tfrac{n\pi r}{b}) Z(r, t)\, dr\, dt \tag{51}$$

$i = 1, 2, \ldots, M$

Making use of (50) and (51), the ith entry in F*FU takes the form

$$f_i^*(\tau; r, t) F(r, t; \alpha)\, U(\alpha) =$$

$$\frac{2}{b} \int_{\tau}^{t_1} \int_{0}^{b} \sum_n e^{\lambda_n (t-\tau)} b_{ni} \sin(\tfrac{n\pi r}{b}) \int_{t_0}^{t} \sum_m e^{\lambda_m (t-\tau)} \sin(\tfrac{m\pi r}{b}) \sum_{j=1}^{M} b_{mn} u_j(\alpha)\, d\alpha\, dr\, dt \tag{52}$$

where

$$b_{ni} = \sin(\tfrac{n\pi r_i}{b}) \sin(\tfrac{\pi r_i}{b}) \phi_M(\tfrac{2}{b}) V \tag{53}$$

and, given the orthogonality property

$$\frac{2}{b}\int_0^b \sin\left(\frac{n\pi r}{b}\right)\sin\left(\frac{m\pi r}{b}\right)dr = \begin{cases} 1 \text{ for } n=m \\ 0 \text{ for } n\neq m \end{cases} \tag{54}$$

the expression (20) becomes

$$f^*(\tau;\, r,\, t)F(r,\, t;\, \alpha)\, U(\alpha) =$$

$$\int_{\tau}^{t_1}\int_{t_0}^{t}\sum_n e^{\lambda_n(2t-\tau-\alpha)}\, b_{ni}\sum_{j=1}^{M} b_{nj}\, u_j(\alpha)\,d\alpha dt \tag{55}$$

which after interchanging the order of integration takes the form

$$f_1^*(\tau;\, r,\, t)\, F(r,\, t;\, \alpha)\, U(\alpha) =$$

$$\int_{t_0}^{t_1}\sum_{j=1}^{M}\sum_n b_{ni}b_{nj}\, K_n(\tau;\, \alpha)\, u_j(\alpha)\,d\alpha \tag{56}$$

where $K_n(\tau;\, \alpha)$ is defined by

$$K_n(\tau;\, \alpha) = \begin{cases} \int_{\tau}^{t_1} e^{\lambda_n(2t-\tau-\alpha)}\,dt & \text{for } \alpha < \tau \\ \int_{\alpha}^{t_1} e^{\lambda_n(2t-\tau-\alpha)}\,dt & \text{for } \alpha \geq \tau \end{cases} \tag{57}$$

and upon integration,

$$K_n(\tau;\alpha)=\begin{cases}[e^{\lambda_n(2t_1-\tau-\alpha)}-e^{\lambda_n|\tau-\alpha|}]/2\lambda_n & \text{for } \lambda_n\neq 0\\ t_1-\tau & \text{for } \lambda_n=0 \text{ and } \alpha<\tau\\ t_1-\alpha & \text{for } \lambda_n=0 \text{ and } \alpha\geq\tau\end{cases} \tag{58}$$

where $|\cdot|$ denotes absolute value.

The function $\Delta(\tau)$ defined in (39) becomes in this case,

$$\Delta(\tau)=-\frac{1}{k_0}F^*(\tau;\ r,\ t)\ G(r,\ t;\ r',\ t_0)\ Z_0(r') \tag{59}$$

and making use of (56) and (49), the ith entry $\Delta_i(\tau)$ is obtained as follows

$$\Delta_i(\tau)=$$

$$-\frac{1}{k_0}\left[\frac{2}{b}\right]\int_\tau^{t_1}\int_0^b\sum_n e^{\lambda_n(t-\tau)}b_{ni}\sin\left(\frac{n\pi r}{b}\right)\sum_m\left[\frac{2}{b}\right]\sin\left(\frac{m\pi r}{b}\right)\Big[$$

$$\int_0^b\sin\left(\frac{m\pi r'}{b}\right)Z_0(r')dr'\Big]e^{\lambda_m(t-t_0)}drdt \tag{60}$$

which reduces to

$$\Delta_i(\tau)=-\frac{1}{k_0}\left[\frac{2}{b}\right]\sum_n\Big[\int_\tau^{t_1}e^{\lambda_n(2t-\tau-t_0)}dt\Big]b_{ni}\Big[\int_o^b\sin\left(\frac{n\pi r'}{b}\right)Z_0(r')dr'\Big] \tag{61}$$

and becomes, after integration,

$$\Delta_i(\tau) =$$

$$\frac{1}{[k_0 b]} \sum_n \frac{[e^{\lambda_n(\tau-t_0)} - e^{\lambda_n(2t_1-t_0-\tau)}]}{\lambda_n} b_{ni} \left[\int_o^b \sin\left(\frac{n\pi r'}{b}\right) Z_0(r')dr'\right] \tag{62}$$

The optimality conditions are finally obtained in the form

$$u_i(\tau) = \Delta_i(\tau) - \frac{1}{k_0} \int_{t_0}^{t_1} \sum_{j=1}^{M} \hat{K}_{ij}(\tau; \alpha)\, u_j(\alpha)d\alpha \tag{63}$$

$$i=1, 2, \ldots, M$$

where $\hat{K}_{ij}$ is defined by

$$\hat{K}_{ij} = \sum_n b_{ni}\, b_{nj}\, K_n(\tau; \alpha) \tag{64}$$

4.3.1 Discussion

More conventional techniques [11] approach this control problem through modal expansion methods and invoke variational principles which yield necessary conditions for optimality in the form of an infinite system of ordinary differential equations with mixed boundary conditions. The question of how many modes should be included in a finite dimensional version of the system is well known to constitute a very difficult problem, which in most cases can only be solved through trial and error. In contrast,

condition (63) forms a system of M integral equations with non-degenerate kernels. Although these kernels are represented by infinite series, the question of how many terms should be considered in a finite series is a tractable problem and, as it is shown below, error bounds can be estimated with relative ease.

Consider the kernel

$$\hat{K}_{ij}(\tau;\ \alpha) = \sum_{n=1} b_{ni}\ b_{nj}\ K_n(\tau;\ \alpha) \tag{64}$$

and the approximate version

$$\tilde{K}_{ij}(\tau;\ \alpha) = \sum_{n=1}^{P} b_{ni}\ b_{nj}\ K_n(\tau;\ \alpha) \tag{65}$$

which is obtained by truncating the series after the first P terms. The error is given by

$$\varepsilon_{ij}(\tau;\ \alpha) = \hat{K}_{ij}(\tau;\ \alpha) - \tilde{K}_{ij}(\tau;\ \alpha)$$

$$= \sum_{n=P+1}^{\infty} b_{ni}\ b_{nj}\ K_n(\tau;\ \alpha) \tag{66}$$

where $K_n(\tau;\ \alpha)$ is defined by (58);

$$K_n(\tau;\ \alpha) = [e^{\lambda_n(2t_1-\tau-\alpha)} - e^{\lambda_n|\tau-\alpha|}]/2\ \lambda_n \tag{67}$$

In view of equation (53), the product $b_{ni}\ b_{nj}$ is bounded absolutely by

$$|b_{ni}\, b_{nj}| \leq \left[\frac{2}{b}\,\phi_M\, V\right]^2 \tag{68}$$

therefore

$$|\varepsilon_{ij}(\tau;\,\alpha)| \leq \left[\frac{2}{b}\right]^2 \phi_M^2\, V^2 \sum_{n=P+1}^{\infty} |K_n(\tau;\,\alpha)| \tag{69}$$

and using (58) it follows that

$$|\varepsilon_{ij}(\tau;\,\alpha)| < \left[\frac{\phi_M V}{b}\right]^2 2 \sum_{n=P+1}^{\infty} \frac{\left[e^{\lambda_n(2t_1-\tau-\alpha)} + e^{\lambda_n|\tau-\alpha|}\right]}{|\lambda_n|} \tag{70}$$

for all subscripts i and j.

Upon integration of the strict inequality (70), the following result is obtained,

$$\int_{t_0}^{t_1}\int_{t_0}^{t_1} |\varepsilon_{ij}(\tau;\,\alpha)|\, d\alpha d\tau <$$

$$\left[\frac{\phi_M V}{b}\right]^2 2 \sum_{n=P+1}^{\infty} \left[\frac{2[t_1-t_0]}{[\lambda_n]^2} + \frac{\left[e^{\lambda_n 2(t_1-t_0)} - 1\right]}{|\lambda_n|^3}\right] \tag{71}$$

Recalling from (47),

$$\lambda_n = [a_1 - n^2]\, a_2 \tag{72}$$

where a_1 and a_2 are the positive real constants

$$a_1 = \frac{[\nu\Sigma_f - \Sigma_a]}{\pi^2 D} \; b^2 \leq 1 \tag{72a}$$

and

$$a_2 = \frac{V \pi^2 D}{b^2} \tag{72b}$$

Making use of the inequality

$$[e^{\lambda_n 2(t_1 - t_0)} - 1] < 1 \tag{73}$$

and substituting (72) in (71), one finds

$$\int_{t_0}^{t_1} \int_{t_0}^{t_1} |\varepsilon_{ij}(\tau;\, \alpha)| \, d\alpha d\tau <$$

$$2\left[\frac{\phi_M V}{b}\right]^2 \sum_{n=p+1}^{\infty} \left[\frac{2[t_1 - t_0]}{[a_2]^2 [n^2-1]^2} + \frac{1}{[a_2]^3 [n^2-1]^3}\right] \tag{74}$$

Replacing the summation sign by the integral symbol and evaluating the integral, the error bound

$$\int_{t_0}^{t_1} \int_{t_0}^{t_1} |\varepsilon_{ij}(\tau;\, \alpha)| \, d\alpha d\tau < \frac{4\phi_M^2 b^2}{\pi^4 D^2} \frac{[t_1 - t_0]}{[P^2+2P]^2} +$$

$$\frac{2\phi_M^2 b^2}{\pi^4 D^2} \left[2[t_1 - t_0] - \frac{3b^2}{4V\pi^2 D}\right] \left[\frac{[P+1]}{2P[P+2]} + \frac{1}{4} \ln \left(\frac{P}{P+2}\right)\right]$$

$$+ \frac{\phi_M^2 b^4}{2V\pi^6 D^3} \left[\frac{[P+1][P^2+2P]+3}{[P^2+2P]^3}\right] \tag{75}$$

is finally obtained. This estimate is a function of P, which represents the number of terms considered in the finite series approximation to $\hat{K}_{ij}(\tau;\ \alpha)$.

4.3.2 A Method for Computing the Optimal Control

Equation (63) can be rewritten in a more compact form

$$U(\tau) = L(\tau;\ \alpha)\ U(\alpha) \tag{76}$$

where $U(\tau)$ is an element of the space H_1 and L is a nonlinear transformation from H_1 into itself defined by

$$L(\tau;\ \alpha)\ U(\alpha) = \Delta(\tau) - \frac{1}{k_0} \int_{t_0}^{t_1} K(\tau;\ \alpha)\ U(\alpha)\ d\alpha \tag{77}$$

and $\Delta(\tau)$ is a function in H_1 with entries $\Delta_i(\tau)$ defined by equation (62). $K(\tau;\ \alpha)$ is a matrix operator with entries $\hat{K}_{ij}(\tau;\ \alpha)$ defined by equation (64).

Invoking the contraction mapping theorem [90], it follows that the sequence

$$U_n(\tau) = L(\tau;\ \alpha)\ U_{n-1}(\alpha) \tag{78}$$

based on the initial guess $U_0(\tau)$ in H_1 would converge to the unique solution to equation (76) provided that L is a contraction, that is, for any two functions U and W in H_1,

$$||LU-LW||_{H_1} \leq C\ ||U-W||_{H_1} \tag{79}$$

where C is a constant and $0<C<1$.

In what follows, a rough estimate of the constant C will be computed in order to test for the contraction condition and establish a criterion for the applicability of the successive approximation algorithm (78) to equation (76).

Consider the norm

$$\left|\left| \int_{t_0}^{t_1} K(\tau;\, \alpha)\, U(\alpha)\, d\alpha \right|\right|_{H_1}^2 \tag{80}$$

and for convenience denote (80) by d. Making use of the definition of $K(\tau;\, \alpha)$ d can be expanded as follows

$$d = \int_{t_0}^{t_1} \int_{t_0}^{t_1} \int_{t_0}^{t_1} \sum_{i=1}^{M} \sum_{j=1}^{M} \sum_{k=1}^{M} \hat{K}_{ij}(\tau;\, \alpha)\, \hat{K}_{ik}(\tau;\, \beta)\, u_j(\alpha)\, u_k(\beta)\, d\alpha d\beta d\tau \tag{81}$$

where u_j is the jth component of U. Substituting the series representation (~~64~~) in (81) and taking the absolute value of the integrand it follows that

$$d \leq \left[\operatorname*{Max}_{ni} b_{ni} \right]^4 M \int_{t_0}^{t_1} \sum_{n=1}^{\infty} \left[\int_{t_0}^{t_1} |K_n(\tau;\, \alpha)| \sum_{j} |u_j(\alpha)|\, d\alpha \right]^2 d\tau \tag{82}$$

which by Schwarz's inequality reduces to

$$d \leq \left[\operatorname*{Max}_{ni} b_{ni} \right]^4 M \left[\sum_{n=1}^{\infty} \int_{t_0}^{t_1} \int_{t_0}^{t_1} [K_n(\tau;\, \alpha)]^2 d\alpha d\tau \right] ||U||_{H_1}^2 \tag{83}$$

In view of the relation (58) it is clear that $K_n(\tau;\alpha)$ is bounded by

$$K_n^2(\tau;\alpha) \leq \begin{cases} [e^{2\lambda_n(2t_1-\tau-\alpha)} + e^{2\lambda_n|\tau-\alpha|}]/4\,[\lambda_n]^2 & \text{for } n>1 \\ [t_1-t_0]^2 & \text{for } n=0 \end{cases} \tag{84}$$

Hence, after integration, one obtains

$$\int_{t_0}^{t_1}\int_{t_0}^{t_1} [K_n(\tau;\alpha)]^2 d\alpha d\tau < [e^{4\lambda_n(t_1-t_0)} - e^{2\lambda_n(t_1-t_0)}$$

$$+ e^{\lambda_n(t_1-t_0)} - 1]/16[\lambda_n]^4 - \frac{[t_1-t_0]}{4[\lambda_n]^3} \quad \text{for } n>1 \tag{85}$$

Since λ_n is negative for all $n>1$, it follows that

$$\int_{t_0}^{t_1}\int_{t_0}^{t_1} [K_n(\tau;\alpha)]^2 d\alpha d\tau < \begin{cases} \dfrac{[t_1-t_0]}{2|\lambda_n|^3} & \text{for } n>1 \\ [t_1-t_0]^4 & \text{for } n=1 \end{cases} \tag{86}$$

and using the definition of λ_n given by (47), (86) reduces to

$$\sum_{n=1}^{\infty} \int_{t_0}^{t_1}\int_{t_0}^{t_1} [K_n(\tau;\alpha)]^2 d\alpha d\tau <$$

$$[t_1-t_0]^4 + \sum_{n=2}^{\infty} \frac{[t_1-t_0]b^6}{(V D\pi^2)^3 2} \left[\frac{1}{(n^2-1)^3}\right] \tag{87}$$

By making use of the upper bound

$$\sum_{n=2}^{\infty} \frac{1}{(n^2-1)^3} = 3.94\text{x}10^{-2} \tag{88}$$

it follows that

$$\left[\sum_{n=1}^{\infty} \int_{t_0}^{t_1} \int_{t_0}^{t_1} [K_n(\tau;\ \alpha)]^2 d\alpha d\tau\right]$$

$$< \left[[t_1-t_0]^4 + 5.77 \times 10^{-3} \frac{[t_1-t_0]b^6}{(V\ D\pi^2)^3}\right] \tag{89}$$

In view of (89) and (83), the following upper bound estimate is finally obtained:

$$\left|\left| \int_{t_0}^{t_1} K(\tau;\ \alpha)\ U(\alpha) d\alpha \right|\right|_{H_1}$$

$$< \left[\frac{2\phi_M V}{b}\right]^2 M^{1/2} [[t_1-t_0]^4 + 1.97\text{x}10^{-2} \frac{[t_1-t_0]b^6}{(V\ D\pi^2)^3}]^{1/2} ||U||_{H_1}$$

(90)

Given that

$$||LU-LW||_{H_1} \leq \frac{1}{k_0} \left|\left| \int_{t_0}^{t_1} K(\tau;\ \alpha)[U(\alpha)-W(\alpha)]d\alpha \right|\right|_{H_1} \tag{91}$$

it follows from (89) that

$$||LU-LW||_{H_1} < C\ ||U-W||_{H_1} \tag{92}$$

where

$$C = \left[\frac{1}{k_0}\right]\left[\frac{2\phi_M V}{b}\right]^2 M^{1/2}[[t_1-t_0]^4 + 1.97 \times 10^{-2}\frac{[t_1-t_0]b^6}{(V D\pi^2)^3}]^{1/2} \tag{93}$$

It is clear from (93) that a sufficiently large value of k_0 would make C less than one.

The design parameter k_0, which was introduced in the performance index (41), provides the flexibility required to obtain the desired balance between system response and control action. A suitable value of k_0 can only be found through systematic searching which of necessity involves repeated computations for different values of k_0.

Making the supposition that the contribution of the flux deviation term at its maximum in the performance index should equal the contribution of the control energy term at its maximum, Wiberg [11] has outlined a method for obtaining an initial rough estimate of k_0. For the case where the flux deviation is to be no more than 10% of the steady state flux and where the control reactivity is to be no more than 10% of the total material reactivity, Wiberg proposed the initial rough estimate

$$k_0 = \frac{2M}{b}\left[\frac{\phi_M}{(\nu\Sigma_f-\Sigma_a)}\right]^2 \tag{94}$$

where M is the number of controls and the other variables are defined as before.

Substituting (94) in (93) the following result is obtained,

$$C = \frac{2}{b}\left[(v\Sigma_f-\Sigma_a)V\right]^2\left[\frac{[t_1-t_0]^4}{M} + 1.97 \times 10^{-2}\ \frac{(t_1-t_0)b^6}{M(V\ D\pi^2)^3}\ \right]^{1/2} \tag{95}$$

Although (95) is a very conservative estimate, it provides a useful criterion to determine whether or not the contraction mapping algorithm would yield a convergent sequence. If C is less than one and U denotes the solution to equation (76) then the contraction mapping theorem ensures that the sequence

$$U_n = L\ U_{n-1} \tag{96}$$

would converge, with the rate of convergence given by

$$||U-U_n||_{H_1} \leq \frac{C^n}{(1-C)}\ ||LU_o-U_o||_{H_1} \tag{97}$$

where U_o is the initial guess. Consider for example the reactor constants given by Wiberg (page 359, reference [11]), $(v\Sigma_f-\Sigma_a)V=0.256\text{sec}^{-1}$, b=250cm, M=2 and VD=1600cm^2/sec. In this case, the criterion (95) would take the form

$$C = 5.24\text{x}10^{-4}[0.5\,[t_1-t_0]^4 + 6.11\text{x}10^{-1}[t_1-t_0]]^{1/2} \tag{98}$$

and since the time scale of the nuclear model considered is of the order of seconds, it can be concluded that the successive approximation algorithm (96) is applicable to problems in this range.

CHAPTER IV

FURTHER APPLICATIONS OF THE MINIMUM NORM FORMULATION TO PROBLEMS IN CONTROL OF DISTRIBUTED REACTORS.

1. Introduction

Several interesting variations on the optimal control problem of Chapter III would result if in addition to the requirement that the control function minimizes a performance index, which takes into account the deviations of the system's state trajectories from a desired distribution, the state of the reactor core is also required to satisfy one of the following conditions:

(a) At the end of a given time-interval the total power output from the reactor core should be equal to a specified value.

This constraint is relevant to the problem of adjusting the power level while minimizing both the control effort and the distortion of the flux distribution.

(b) The total power output from the reactor core should match a specified load trajectory along a given time-interval.

This constraint is relevant to the problem of controlling a nuclear reactor core during load-following operations.

(c) At the end of the specified time-interval the state of the system should match a given distribution.

Although this constraint is also relevant to the problem of adjusting the power output from the core, it is more restrictive than case (a), stated above.

The present chapter is devoted to the application of the minimum norm formulation to these particular problems. The scope of the chapter

is focussed upon the derivation of necessary and sufficient conditions for optimality in a form that lends itself to computation.

2. The Nuclear Reactor Model

Although an accurate description of the power generated in a nuclear reactor core would have to take into account both the continuous energy spectra of the neutrons in the core and the energy-dependent fission cross section of the fuel, adequate results are often obtained through the simplifying assumption that the neutron population can be grouped into a number of monoenergetic groups whose dynamics are described by multigroup diffusion theory [37].

Here it is assumed that the total power produced in a nuclear reactor core is given by

$$p(t) = \sum_{i=1}^{N_g} \int_V e_{f_i} \Sigma_{f_i}(r)\, \phi_i(r, t)\, dr \qquad (1)$$

where the integral is over the volume of the reactor core and e_{f_i} is the energy released per fission involving a neutron in the ith-energy flux ϕ_i. Σ_{f_i} is the fission macroscopic cross section for neutrons in the ith-energy interval. N_g denotes the number of energy intervals which cover the neutron energy range.

This assumption is often employed in engineering applications. The reader is referred to Chapter IV of reference [91] for a detailed discussion of the one-energy group approximation.

As was the case in Chapter III of this thesis, here the assumption is made that near an equilibrium condition the state, $\psi(r, t)$, of the nuclear reactor core is given by

$$\psi(r, t) = G(r, t; r', t_0) Z_0(r') + \int_{t_0}^{t} G(r, t; r', \tau)\, B(r')\, U(\tau)\, d\tau \qquad (2)$$

where all the variables are defined as before.

In terms of the state $\psi(r, t)$ the total power $p(t)$ can be modelled by the expression

$$p(t) = \int_V \hat{H}(r)\, \psi(r, t)dr \tag{3}$$

where $\hat{H}(r)$ is a space-dependent row matrix with the appropriate dimension.

In what follows it is assumed that the state $\psi(r, t)$ and the control $U(t)$ belong to the Hilbert spaces H_2 and H_1 respectively. H_1 and H_2 are endowed with the inner products

$$\langle \psi, Z \rangle_{H_2} = \int_{t_0}^{t_1} \langle \psi(t), Z(t) \rangle_H dt \tag{4}$$

and

$$\langle U, W \rangle_{H_1} = \int_{t_0}^{t_1} k_0 \langle U(t), W(t) \rangle_E dt \tag{5}$$

where k_0 is a positive real number. The inner products in H and E are defined by

$$\langle \psi(t), Z(t) \rangle_H = \int_V Z^T(r, t)\, Q(r)\, \psi(r, t)dr \tag{6}$$

and

$$\langle U(t), W(t) \rangle_E = W^T(t)\, R\, U(t) \tag{7}$$

in which R is a M-dimensional positive definite matrix. Q(r) is a space-dependent, N-dimensional positive definite matrix.

3. Optimal Control of the State Distribution with Power Level Adjustment

Suppose that at time t_0 the total power output corresponding to the initial state $Z_0(r)$ is p_0. The problem of finding the control function $U(t)$ that changes the total power output from p_0 to p_1 in a given time interval $[t_0, t_1]$, and minimizes the performance index

$$J(U) = \int_{t_0}^{t_1} \int_V [\psi(r, t) - Z(r, t)]^T Q(r)[\psi(r, t) - Z(r, t)]drdt$$

$$+ k_0 \int_{t_0}^{t_1} U^T(t) R U(t)dt \qquad (8)$$

where $Z(r, t)$ is the desired state distribution, will be considered in this section.

It should be noticed that although no explicit constraint is imposed on the total power output

$$p(t) = \int_V \hat{H}(r) G(r, t; r', t_0) Z_0(r')dr$$

$$+ \int_V \int_{t_0}^{t_1} \hat{H}(r) G(r,t; r', \tau) B(r') U(\tau)d\tau dr \qquad (9)$$

along the given time interval, implicitly, the performance index (8) does penalize the output power variations along the trajectory.

The present problem includes on an equal footing the problem of controlling flux oscillations, minor power adjustments and load following operations. The particularization of the problem into any of these cases depends on both the time scale and the reactor model considered.

In order to be able to profit from the results discussed in Chapter II, one should consider first the formulation of the problem in the context of functional anaylsis. With this in mind the following definitions are introduced: F denotes the transformation from H_1 into H_2 defined by

$$F(r, t;\tau)\, U(\tau) = \int_{t_0}^{t} G(r, t; r', \tau)\, B(r')\, U(\tau) d\tau \,. \qquad (10)$$

T denotes the transformation

$$T(t; \tau)\, U(\tau) = \int_{t_0}^{t} \int_{V} \hat{H}(r)\, G(r, t; \tau)\, B(r')\, U(\tau) dr d\tau \,. \qquad (11)$$

Also, $y(r, t)$ and $\xi(t)$ are functions defined by

$$y(r, t) = Z(r, t) - G(r, t; r', t_0) Z_0(r') \qquad (12)$$

and

$$\xi(t) = p(t) - \int_{V} \hat{H}(r)\, G(r, t; r', t_0)\, Z_0(r') dr \qquad (13)$$

It should be noticed that at time t_1 the function $\xi(t_1)$ is completely specified. The transformation T and the function ξ when evaluated at time t_1 will be denoted by T_1 and ξ_1. Finally, H_3 represents the one-dimensional real Hilbert space endowed with the inner product

$$\langle \alpha, \beta \rangle_{H_3} = \alpha\beta \qquad (14)$$

With these definitions the optimal control problem of this section can be formulated as follows:

Find the control U in H_1 that minimizes

$$J(U) = ||FU-y||_{H_2} + ||U||_{H_1} \qquad (15)$$

and satisfies

$$\xi_1 = T_1 U \qquad (16)$$

where T_1 is a linear and bounded transformation from H_1 onto H_3.

This formulation is recognized at once as Porter's abstract minimum norm problem.

3.1 The Necessary and Sufficient Conditions of Optimality

By invoking Porter's result it follows that the control U which minimizes the performance index (15) and satisfies (16) is given by

$$U = [I + F^*F]^{-1} (T_1^+ \eta + F^*y) \qquad (17)$$

where η is the unique element of H_3 satisfying

$$T_1 U = \xi_1 \, . \qquad (18)$$

The adjoint F^* transforms H_2 into H_1. The pseudoinverse T_1^+, associated with T_1, is given by

$$T_1^+ = T_1^*[T_1 T_1^*]^{-1} \tag{19}$$

and the adjoint T_1^* is a transformation which maps H_3 into H_1.

Since there is no straightforward approach to computing the inverse of $[I + F^*F]$ it is necessary to express conditions (17) and (18) in a more standard form, amenable to computation. To this end, condition (17) is rewritten as follows

$$U = - F^* F U + T_1^+ \eta + F^* y \ . \tag{20}$$

Also, given the algebraic property of the pseudoinverse

$$T_1 T_1^+ \eta = \eta \tag{21}$$

and making use of (18), one finds from (20) that

$$\xi = - T_1 F^* F U + \eta + T_1 F^* y \ . \tag{22}$$

Solving for η from (22) and substituting it in (20), one obtains the following operator equation

$$U = [T_1^+ T_1 - I] F^* F U + \Delta \tag{23}$$

in which the function Δ is an element of H_1 defined by

$$\Delta = T_1^+ \xi - [T_1^+ T_1 - I] F^* y \tag{24}$$

The reader may notice that (23) lends itself to application of successive approximation techniques. It is true that the pseudoinverse T_1^+ implicitly involves the inverse of the operator $T_1 T_1^*$. However, given the fact that T_1 has finite dimensional range, it follows that $T_1 T_1^*$ is an operator with finite dimensional domain and range. More explicitly, $T_1 T_1^*$ transforms the one-dimensional Hilbert space H_3 into itself. The inverse $[T_1\ T_1^*]^{-1}$, therefore, should pose no computational difficulties.

In order to fully characterize the optimality condition (23), it is still necessary to derive explicit expressions for the adjoint F^* and the pseudoinverse T^+.

3.2 The Adjoint F^*

Following the procedure outlined in section 4.1 of Chapter III, it can be shown that F^* is given by

$$F^*(\tau,\ t)\ Z(t) = k_0^{-1} \int_{\tau}^{t_1} B^* G^*(t;\ \tau)\ Z(t)dt \qquad (25)$$

where for simplicity only the temporal variable is explicitly shown. The adjoint B^* transforms H into E according to

$$B^* Z = R^{-1} \int_V B^T(r)\ Q^T(r)\ Z(r)dr \qquad (26)$$

and the adjoint G^* transforms H into itself.

Replacing Z(t) in equation (25) by the defining relation (10), one obtains

$$F^*(\tau;\ t)\ F(t;\ \alpha)\ U(\alpha) = k_0^{-1} \int_{\tau}^{t_1} B^* G^*(t;\ \tau) \int_{t_0}^{t} G(t;\ \alpha)\ B\ U(\alpha)d\alpha dt \qquad (27)$$

where again for simplicity only the temporal dependency is explicitly shown.

After interchanging the order of integration in expression (27), it follows that

$$F^*(\tau;\, t)\, F(t;\, \alpha)\, U(\alpha) = k_Q^{-1} \int_{t_0}^{t_1} K(\tau;\, \alpha)\, U(\alpha)\, d\alpha \qquad (28)$$

where

$$K(\tau;\, \alpha) = \begin{cases} \int_{\tau}^{t_1} B^* G^*(t;\, \tau)\, G(t;\, \alpha)\, B\, dt & \text{for } \alpha < \tau \qquad (29) \\ \int_{\alpha}^{t_1} B^* G^*(t;\, \tau)\, G(t;\, \alpha)\, B\, dt & \text{for } \alpha \geq \tau \qquad (30) \end{cases}$$

3.3 The Pseudoinverse T_1^+

The adjoint T_1^* is related to T_1 through the inner product relation

$$\langle T_1\, U,\, \xi \rangle_{H_3} = \langle U,\, T_1^*\, \xi \rangle_{H_1} . \qquad (31)$$

Using the expression for T_1 given by (11), the left side of (31) is expanded as follows

$$\int_{t_0}^{t_1} \int_V \hat{H}(r)\, G(r,\, t_1;\, r',\, \tau)\, B(r')\, U(\tau)\, dr\, d\tau\, \xi \qquad (32)$$

which, in terms of the inner product in H, takes the form

$$\int_{t_0}^{t_1} \langle G(t_1;\, \tau)\, B\, U(\tau),\, Q^{-1} \hat{H}^T \xi \rangle_H\, d\tau . \qquad (33)$$

From (33) it follows that

$$\langle T_1 U, \xi \rangle_{H_3} = \langle U, k_0^{-1} B^* G^* Q^{-1} \hat{H}^T \xi \rangle_{H_1} . \tag{34}$$

The adjoint T_1^* is therefore given by

$$T_1^*(\tau) = k_0^{-1} B^* G^*(t_1; \tau) Q^{-1} \hat{H}^T . \tag{35}$$

Upon substitution of the expression for B^*, given in (26), (35) leads to

$$T_1^*(\tau) = k_0^{-1} R^{-1} \int_V B^T(r) Q^T(r) G^*(r, t_1; r', \tau) Q^{-1}(r') \hat{H}^T(r') dr \tag{36}$$

In view of (36), the term $T_1 T_1^*$ becomes

$$T_1 T_1^* = d\, k_0^{-1} \tag{37}$$

where d is the real number defined by

$$d = \int_{t_0}^{t_1} \int_V \int_V [\hat{H}(r) G(r, t_1; r', \tau) B(r') R^{-1} B^T(\beta) Q^T(\beta)]$$

$$\times [G^*(\beta, t_1; r'', \tau) Q^{-1}(r'') \hat{H}^T(r'')]\, d\beta\, dr\, d\tau \tag{38}$$

It follows from (36) and (37) that the pseudoinverse $T_1^*[T_1 T_1^*]^{-1}$ can be finally obtained in the form

$$T_1^+(\tau) = d^{-1} K_I(\tau) \tag{39}$$

where $K_I(\tau)$ is an M-dimensional column matrix defined by

$$K_I(\tau) = R^{-1} \int_V B^T(r)\, Q^T(r)\, G^*(r, t_1; r', \tau)\, Q^{-1}(r')\, \hat{H}^T(r')dr \tag{40}$$

3.4 The Optimal Control: The Solution to a Fredholm's Integral Equation

With the help of the explicit relations for F^* and T_1^+, it is now possible to fully characterize the optimality condition (23).

Recalling expression (11) for T_1 and expression (28) for F^*FU one obtains

$$T_1 F^* FU = k_0^{-1} \int_{t_0}^{t_1} K_{II}(\alpha)\, U(\alpha)\, d(\alpha) \tag{41}$$

where the kernel $K_{II}(\alpha)$ is defined by

$$K_{II}(\alpha) = \int_{t_0}^{t_1} \int_V \hat{H}(r)\, G(r, t_1; r', \tau)\, B(r')\, K(\tau; \alpha) dr d\tau \tag{42}$$

and, in view of (39), it follows that

$$T_1^+(\tau)\, T_1 F^* FU = (d\, k_0)^{-1} \int_{t_0}^{t_1} K_I(\tau)\, K_{II}(\alpha)\, U(\alpha)\, d\alpha \,. \tag{43}$$

Finally, making use of (43) and (28), one finds that the optimality condition can be reformulated in terms of a nonhomogeneous Fredholm's integral equation of the second kind,

$$U(\tau) = \Delta(\tau) + k_0^{-1} \int_{t_0}^{t_1} [d^{-1} K_I(\tau)\, K_{II}(\alpha) - K(\tau;\, \alpha)]\, U(\alpha)\, d\alpha \tag{44}$$

where $\Delta(\tau)$ is the function defined by (24).

It would be interesting to point out certain differences between this and the corresponding result obtained in Chapter III. The reader may recall that the present problem differs from the case treated in Chapter III only in that the present problem considers the additional requirement that the total power output should match a specified level p_1 at time t_1.

Comparing equation (44) and equation (38) of Chapter III, which constitute the optimality conditions for both problems, it can be noticed that the additional requirement manifests itself in the form of two changes introduced in equation (38). The first is the addition of a new term, $d^{-1} K_I(\tau)\, K_{II}(\tau)$, to the kernel $K(\tau;\, \alpha)$ of equation (III, 38). The second effect is the addition of $T_1^+ \xi - T_1^+ T_1 F^* y$ to the forcing function $\Delta(\tau)$ in equation (III, 39).

The advantages of the present optimization approach over more conventional techniques are self evident. Techniques based on variational principles and modal expansion methods would yield necessary conditions for optimality in the form of an infinite system of ordinary differential equations with mixed boundary conditions. In contrast, the necessary and sufficient condition (44) constitutes a finite system of integral equations, with as many equations as there are control devices in the reactor core.

Equation (44) is amenable to application of successive approximation

techniques and, as it was the case in Chapter III, a convergence analysis of the contraction mapping algorithm could be carried out with relative ease.

3.5 An Example

Consider the slab reactor model of the problem example given in section 4.3 of Chapter III.

In the neighborhood of the equilibrium distribution $\phi_0(r)$,

$$\phi_0(r) = \phi_M \sqrt{\frac{2}{b}} \sin(\frac{\pi r}{b}) \tag{45}$$

the state deviation $\psi(r, t)$, at any time $t \geq t_0$, is determined by

$$\psi(r, t) = G(r, t; r', t_0)\, Z_0(r') + F(r, t; \tau)\, U(\tau) \tag{46}$$

where

$$G(r, t; r', t_0)\, Z_0(r') =$$

$$\sum_n [\, \frac{2}{b}\,] \sin(\frac{n\pi r}{b}) [\int_0^b \sin(\frac{n\pi r'}{b})\, Z_0(r')dr'] e^{\lambda_n (t-t_0)} \tag{47}$$

and

$$F(r, t; \tau)\, U(\tau) = \tag{48}$$

$$[\, \frac{2}{b}\,]^{3/2} \phi_M V \int_{t_0}^{t} \sum_n e^{\lambda_n (t-\tau)} \sin(\frac{n\pi r}{b}) [\sum_{i=1}^{M} \sin(\frac{n\pi r_i}{b}) \sin(\frac{\pi r_i}{b}) \mu_i(\tau)]d\tau.$$

$Z_0(r)$ is the state deviation at time t_0 and the eigenvalues λ_n are defined by

$$\lambda_n = \left[\frac{(v\Sigma_f - \Sigma_a)b^2}{\pi^2 D} - n^2\right] \frac{\pi^2 DV}{b^2} . \tag{49}$$

Associated with $Z_0(r)$ there is a total power output p_0, given by,

$$p_0 = e_f \Sigma_f \int_0^b Z_0(r)\, dr \tag{50}$$

If in addition to the requirement of the problem in section 4.3 in which the control $U(\tau)$ minimizes

$$J(U) = \int_{t_0}^{t_1} \int_0^b \psi^2(r, t)\, drdt + k_0 \int_{t_0}^{t_1} U^T(\tau)\, U(\tau) d\tau \tag{51}$$

the control function is also required to satisfy the power constraint

$$p_1 = 0 = e_f \Sigma_f \int_0^b \psi(r, t_1) dr \tag{52}$$

then it follows from the previous discussion that the operators T_1, T_1^+ and T_1^* constitute the only new concepts required to fully characterize the necessary and sufficient condition for optimality. The particularization of these operators to the present case is given below.

In view of (52) and (11) it follows that

$$T_1 U = e_f \Sigma_f \int_0^b \int_{t_0}^{t_1} G(r, t_1; r', \tau) B(r') U(\tau) d\tau dr$$

$$= e_f \Sigma_f \int_0^b F(r, t_1; \tau) U(\tau) d\tau . \tag{53}$$

Furthermore, making use of (48) one finds that

$$T_1 U =$$

$$e_f \Sigma_f \frac{4\phi_M V}{\pi} \left[\frac{2}{b}\right]^{1/2} \int_{t_0}^{t_1} \sum_{n=1}^{\infty} \sum_{i=1}^{M} \frac{e^{\lambda_{2n-1}(t_1-\tau)}}{(2n-1)} \sin\left(\frac{(2n-1)\pi r_i}{b}\right) \sin\frac{(\pi r_i)}{b} u_i(\tau) d\tau \tag{54}$$

It may be recalled from section 4.3 that B(r) denotes the M-dimensional row matrix operator with entries

$$V\phi_M \sqrt{\frac{2}{b}} \sin\left(\frac{\pi r}{b}\right) \delta(r-r_i) \tag{55}$$

Given that $G(r, t; r', \tau)$ is self-adjoint and in view of (36), it follows that the adjoint T_1^* is an M-dimensional, time-dependent column matrix with entries $T_{1_i}^*(\tau)$ given by

$$T_{1_i}^*(\tau) =$$

$$e_f \Sigma_f \frac{4\phi_M V}{\pi k_0} \left[\frac{2}{b}\right]^{1/2} \sin\left(\frac{\pi r_i}{b}\right) \sum_{n=1}^{\infty} \sin\left(\frac{(2n-1)\pi r_i}{b}\right) \frac{e^{\lambda_{2n-1}(t_1-\tau)}}{(2n-1)} \tag{56}$$

Making use of (54) and (56), one finds

$$T_1 T_1^* = \left[\frac{e_f \Sigma_f \phi_M V}{\pi}\right]^2 \frac{32}{bk_0} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{i=1}^{M}$$

$$\sin\left(\frac{(2n-1)\pi r_i}{b}\right) \sin\left(\frac{(2m-1)\pi r_i}{b}\right) \sin^2\left(\frac{\pi r_i}{b}\right)$$

$$\left[\frac{e^{(\lambda_{2n-1}+\lambda_{2m-1})(t_1-t_0)} - 1}{(2n-1)(2m-1)(\lambda_{2n-1}+\lambda_{2m-1})} \right] \qquad (57)$$

Finally, it follows from $T_1^+ = T_1^*[T_1 T_1^*]^{-1}$ that the pseudo inverse T_1^+ is a M-dimensional column matrix with entries $T_{1_i}^+(\tau)$ given by

$$T_{1_i}^+(\tau) = \frac{T_{1_i}^*(\tau)}{[T_1 T_1^*]} \qquad (58)$$

It can be shown that the kernel $[d^{-1}K_I(\tau)K_{II}(\alpha) - K(\tau; \alpha)]$ in the optimality condition, for this particular example is an M-dimensional matrix with entries $\bar{K}_{\ell j}(\tau; \alpha) - \hat{K}_{\ell j}(\tau; \alpha)$ in the ℓth-row and jth-column given by

$$\bar{K}_{\ell j}(\tau; \alpha) = C_0^{-1}\left[\frac{\phi_M 2V}{b}\right]^2 \sum_{p=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{i=1}^{M}$$

$$e^{\lambda_{2p-1}(t_1-\tau)} (2p-1)^{-1} (2n-1)^{-1} \sin\left(\frac{(2n-1)\pi r_i}{b}\right)$$

$$\sin^2\left(\frac{\pi r_i}{b}\right) \sin\left(\frac{(2p-1)\pi r_\ell}{b}\right) \sin\left(\frac{\pi r_\ell}{b}\right) \sin\left(\frac{m\pi r_i}{b}\right)$$

$$\sin\left(\frac{m\pi r_1}{b}\right) \sin\left(\frac{\pi r_i}{b}\right) \int_{t_0}^{t_1} e^{\lambda_{2n-1}(t_1-\beta)} K_m(\beta; \alpha) d\beta \qquad (59)$$

where

$$C_0 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{i=1}^{M} \sin\left(\frac{(2n-1)\pi r_i}{b}\right) \sin\left(\frac{(2m-1)\pi r_i}{b}\right)$$

$$\sin^2\left(\frac{\pi r_i}{b}\right) \left[e^{(\lambda_{2n-1} + \lambda_{2m-1})(t_1 - t_0)} - 1\right]$$

$$(2n-1)^{-1} \, (2m-1)^{-1} \, (\lambda_{2n-1} + \lambda_{2m-1})^{-1} \qquad (60)$$

and

$$K_m(\tau; \alpha) = \begin{cases} \left[e^{\lambda_m(2t_1 - \tau - \alpha)} - e^{\lambda_m|\tau - \alpha|}\right](2\lambda_m)^{-1} & \text{for } \lambda_m \neq 0 \\ t_1 - \tau & \text{for } \lambda_m = 0 \text{ and } \alpha < \tau \\ t_1 - \alpha & \text{for } \lambda_m = 0 \text{ and } \alpha \geq \tau \end{cases} \qquad (61)$$

Also,

$$\hat{K}_{\ell j}(\tau; \alpha) = \left[\frac{2\phi_M V}{b}\right]^2 \sum_{m=1}^{\infty} \sin\left(\frac{m\pi r_\ell}{b}\right) \sin\left(\frac{m\pi r_j}{b}\right)$$

$$\sin\left(\frac{\pi r_\ell}{b}\right) \sin\left(\frac{\pi r_j}{b}\right) K_m(\tau; \alpha) \qquad (62)$$

Computation of the optimal control would require that the kernel

in the optimality condition be replaced by an approximate version, which for example, could be obtained by truncating the series in the defining relations (59) and (62). In addition, the convergence rate of the successive approximation algorithm would have to be analyzed.

Although the present case is more involved, the derivation of error estimates between the true and the approximate kernels, and also the computation of the convergence rate of the contraction mapping algorithm, could be carried out in the same manner as it was done in Chapter III.

4. Some Comments on the Problem of Controlling the State Distribution During Load-following

A natural variation on the previous problem would arise if the requirement that the total power output from the reactor core matches a specified level at time t_1 is replaced by the more stringent constraint that the total power output should match a given load trajectory at a finite number of points, in the time-interval of interest. This formulation, clearly, has relevance to the problem of controlling the reactor core during load following operations. Although the previous and the present problem can be treated on an equal footing by the optimization technique of the minimum norm, the latter is, of course, more involved computationally.

It may be recalled from the previous section that the relation connecting the control function U and the total power output $p(t_k)$, at time t_k, is given by

$$\xi(t_k) = T(t_k; \tau)\, U(\tau) \tag{63}$$

where $\xi(t_k)$ is defined by

$$\xi(t_k) = p(t_k) - \int_V \hat{H}(r) G(r, t_k; r', t_0)\, Z_0(r')dr \tag{64}$$

and $T(t_k; \tau)$ defines the transformation from H_1 into the real line,

$$T(t_k; \tau) = \int_V \hat{H}(r)\, F(r, t_k; \tau)\, U(\tau)dr$$

$$= \int_{t_0}^{t} \int_V \hat{H}(r)\, G(r, t_k; r', \tau)\, B(r')\, U(\tau)drd\tau \tag{65}$$

It is clear that if the power output is specified at the finite set of points $\{t_k\}$; k=1, 2, ... N_k, along the time-interval $[t_0, t_1]$, then the function $\xi(t_k)$ would also be completely specified in $\{t_k\}$.

One of the typical features that characterize the optimization techniques of functional analysis is that the transformations involved in the formulation of the problem are defined on normed function spaces.

In the present case, the function $\xi(t_k)$; k=1, 2, ... N_k defined in (64) can be seen to be an element of a finite-dimensional linear space. In fact, it could be considered to be an element of the Hilbert space H_4, (an N_k-dimensional Euclidean space) with inner product

$$< \xi, p >_{H_4} = \sum_{k=1}^{N_k} \xi(t_k)\, p(t_k) . \qquad (66)$$

If ξ denotes the column vector in H_4 with entries $\xi(t_k)$ defined by (63) and $\hat{T}$ represents the column matrix transformation with entries $T(t_k; \tau)$ defined by (65) then the present control problem could now be reformulated in the context of functional analysis.

Find the control $U \varepsilon H_1$ that minimizes

$$J(U) = ||FU-y||_{H_2} + ||U||_{H_1} \qquad (67)$$

and satisfies

$$\xi = \hat{T} U . \qquad (68)$$

Again, it follows from Porter's result that the necessary and

sufficient conditions for optimality are given by

$$U = - F^{*}FU + \hat{T}^{+}\eta + F^{*}y \tag{69}$$

and

$$\xi = - \hat{T}F^{*}FU + \eta + \hat{T}F^{*}y \tag{70}$$

where now η is an element of H_4 and $\hat{T}^{+}$ is the pseudoinverse of $\hat{T}$.

It should be noticed that in this case the pseudoinverse involves the inversion of the N_k-dimensional matrix defined by $\hat{T}\,\hat{T}^{*}$. Where as usual, the asterisk denotes the adjoint.

For N_k small, the computation of $\hat{T}^{+}$ poses no problems and the conditions (69) and (70) may be combined to yield the operator equation

$$U = [\hat{T}^{+}\hat{T} - I]\, F^{*}FU + \Delta \tag{71}$$

in which the function Δ is defined by

$$\Delta = \hat{T}^{+}\xi - [\hat{T}^{+}\hat{T} - I]\, F^{*}y \tag{72}$$

However, for large values of N_k the computation of $[\hat{T}\hat{T}^{*}]^{-1}$ becomes more difficult and some suitable iterative method would have to be used. For this reason it is convenient to rewrite conditions (69) and (70) in a form that does not involve, explicitly, the pseudoinverse. To this end, the function W in H_4 is defined as follows

$$\hat{T}\,\hat{T}^{*}\,W = \eta \tag{73}$$

Substituting (73) in (69) and (70) the desired result is obtained:

$$U = -F^{*}\,F\,U + \hat{T}^{*}W + F^{*}y \tag{74}$$

and

$$0 = -\hat{T}\,F^{*}F\,U + \hat{T}\,\hat{T}^{*}W + \hat{T}\,F^{*}y - \xi \tag{75}$$

Finally, introducing the parameter ε in order to rewrite (75) in a form that lends itself to application of successive iteration techniques,

$$W = \varepsilon\,\hat{T}\,F^{*}\,F\,U - [\varepsilon\,\hat{T}\,\hat{T}^{*} - I]\,W + \varepsilon[\xi - TF^{*}y] \tag{76}$$

in which the parameter ε may be used to improve the rate of convergence of the iterative algorithm employed.

It is interesting to point out that in the limiting case where N_k tends to infinity, the Hilbert space H_4 would no longer remain finite-dimensional. The inner product would be defined by

$$\langle \xi, p \rangle_{H_4} = \int_{t_0}^{t_1} \xi(t)\,p(t)\,dt \tag{77}$$

and the transformation $\hat{T}$ would simply become T, as defined in (11). The optimality conditions would be given in this case by expressions similar to (74) and (76) where instead of $\hat{T}$, the transformation $\hat{T}$ is

written. It is not difficult to show that in this limiting case, the adjoint T^* is given by

$$T^*(\tau;\, t)\, \xi(t) =$$

$$k_0^{-1} \int_{\tau}^{t_1} R^{-1} \int_{V} B^T(r) Q^T(r) G^*(r,\, t;\, r',\, \tau) Q^{-1}(r') H^T(r')\, dr\, \xi(t)\, dt \tag{78}$$

This section is ended with one last but important observation. The reader should notice that the problem discussed above is based on the implicit assumption that the specified load trajectory lies in the range of the transformation T. This assumption is of course required in order to ensure that the equation

$$T\, U = \xi \tag{79}$$

has at least one solution.

5. Optimal Control of the State Distribution with Fixed End State

Another variation on the problem of controlling the state distribution along a given time-interval would arise if in addition to the requirement that the control function U should minimize the functional

$$J(U) = ||FU-y||_{H_2} + ||U||_{H_1} , \tag{15}$$

the requirement that the state should reach a specified distribution $Z_1(r)$ at time t_1 is also considered. It is clear that this formulation would contain the case treated in section 3, in which only the total power output is required to match a given power level at the end of the time-interval. The present problem, however, would be much more restrictive and consequently much more involved computationally.

The additional restriction imposed on U would take in this case the form,

$$Z_1(r) - G(r, t_1; r', t_0)\, Z_0 = F(r, t_1; \tau)\, \bar{U}(\tau) \tag{80}$$

where all the variables are defined as before. The optimality conditions would be given by

$$U = - F^* FU + F_1^* W + F^* y \tag{81}$$

and

$$W = \varepsilon F_1 F^* FU - [\varepsilon F_1 F_1^* - I]\, W + \varepsilon[\xi - F_1 F^* y] \tag{82}$$

where F_1 denotes the operator $F(r, t_1; \tau)$, mapping H_1 into H. W is an unknown function in H and ξ is a function defined by

$$\xi(r) = Z_1(r) - G(r, t_1; r', t_0)\, Z_0 \tag{83}$$

The adjoint F_1^* is given by

$$F_1^*(\tau; r')W(r') = k_0^{-1} R^{-1} \int_V B^T(r) Q^T(r) G^*(r, t_1; r', \tau)\, W(r')dr \tag{84}$$

and the other variables in (80) and (82) are defined as before.

Although the conditions (81) and (82) are amenable to the application of successive iteration techniques, the degree of complexity posed by these conditions is formidable. In order to show how difficult it would be to implement an iterative algorithm, it suffices to realize that at each iteration step, in addition to the trial control function, at least one state distribution defined throughout the reactor core and the time interval $[t_0, t_1]$ would have to be stored in memory. This requirement severely limits the application of the present problem formulation.

The degree of computational complexity that characterizes this problem could be reduced considerably if the requirement that the state of the reactor core should match a specified distribution at time t_1 is replaced by the less restrictive requirement that only a projection of the state should match a given distribution in a finite-dimensional subspace of H at time t_1. Specifically, if the state distribution $\psi(r, t_1)$ is represented in terms of a complete basis $\{\psi_n\}$ in H,

$$\psi(r, t_1) = \sum_{n=1}^{\infty} x_n(t_1)\, \psi_n(r) \tag{85}$$

where $\{x_n\}$ is the set of expansion coefficients, then the formulation of the less restrictive requirement would take the following mathematical form:

$$x_n(t_1) - \langle G(t_1; t_0)\, Z_0, \psi_n^* \rangle_H = \langle F(t_1; \tau)\, U(\tau), \psi_n^* \rangle_H \tag{86}$$

$$n=1, 2, \ldots, M_1$$

where $\{\psi_n^*\}$ is the associated dual set of $\{\psi_n\}$ satisfying

$$\langle \psi_m, \psi_n^* \rangle_H = \delta_{mn} \tag{87}$$

and M_1 is the dimension of the subspace of H into which the state is projected.

It is clear that in the limit as M_1 tends to infinity the requirement (86) would become identical to (80). For the particular case, for example, where $\{\psi_n\}$ and $\{\psi_n^*\}$ are chosen to be the natural modes, described by Kaplan in [7], and which satisfy the relations

$$A\, \psi_n = \lambda_n \psi_n \tag{88}$$

and

$$A^* \psi_n^* = \lambda_n \psi_n^* \tag{89}$$

where A is the operator that appears in the state equation

$$\frac{\partial \psi}{\partial t}(r, t) = A(r)\, \psi(r, t) + B(r)\, U(t)\ , \tag{90}$$

the coefficients $x_n(t)$ of the expansion

$$\psi(r, t) = \sum_{n=1}^{\infty} x_n(t)\, \psi_n(r) \tag{91}$$

would be given by

$$x_n(t) = e^{\lambda_n (t-t_0)} \langle Z_0, \psi_n^* \rangle_H + \sum_{j=1}^{M} \int_{t_0}^{t} e^{\lambda_n (t-\tau)} \langle B_j, \psi_n^* \rangle_H u_j(\tau)\, d\tau \tag{92}$$

where B_j is the jth-column of B and u_j is the ith-entry in the control vector U. Also, the linear constraint in (86) would take the form:

$$x_{1_n} - e^{\lambda_n (t_1 - t_0)} \langle Z_0, \psi_n^* \rangle_H = \sum_{j=1}^{M} \int_{t_0}^{t_1} e^{\lambda_n (t-\tau)} \langle B_j, \psi_n^* \rangle_H u_j(\tau)\, d\tau\ . \tag{93}$$

$n=1, 2, \ldots, M_1$

where the $\{x_{1_n}\}$ are the coefficients in the series representation of $Z_1(r)$.

5.1 Suboptimal Control

So far we have not directed our attention to problems in which the space of controls is finite dimensional. This situation arises naturally when digital computers constitute the main controller in a feedback system, or artificially when either time-discretization or function-expansion techniques are applied in order to simplify a particular optimal control problem.

In what follows, it is assumed that the control function is given by a finite series expansion in terms of a set of known functions of time $\omega_j(t)$, $j=1, 2, \ldots, M_2$ that is

$$u_i(t) = \sum_{j=1}^{M_2} \alpha_{ij}\, \omega_j(t) \tag{94}$$

where $\{\alpha_{ij}\}$ are the expansion coefficients.

It may be noticed that this concept applies equally well to the cases where the control function is either discrete or continuous. The particularization to either case depends, obviously, on the nature of the functions $\{\omega_j\}$.

In view of the modal expansions (91) and (94), it follows that the cost function (8),

$$J(U) = \int_{t_0}^{t_1} \int_V [\psi(r, t)-Z(r, t)]^T Q(r)[\psi(r, t)-Z(r, t)]\,dr\,dt$$

$$+ k_0 \int_{t_0}^{t_1} U^T(t) R\, U(t)\,dt \tag{8}$$

can be rewritten in the form:

$$J(U) = \int_{t_0}^{t_1} \sum_{j=1}^{} \sum_{m=1}^{} (x_j(t) - z_j(t))\, \rho_{jm} (x_m(t) - z_m(t))dt$$

$$+ k_0 \sum_{n=1}^{M} \sum_{i=1}^{M} \sum_{j=1}^{M_2} \sum_{k=1}^{M_2} \alpha_{nj}\, \beta_{njki}\, \alpha_{ik} \qquad (95)$$

where $z_j(t)$ is the jth-expansion coefficient of the desired distribution $Z(r, t)$. ρ_{jm} is given by

$$\rho_{jm} = \int_V \psi_j^T(r)\, Q(r)\, \psi_m(r)dr \,. \qquad (96)$$

Also,

$$\beta_{njki} = \int_{t_0}^{t_1} \omega_j(t)\, r_{ni}\, \omega_k(t)dt \qquad (97)$$

where r_{ni} is the ni-th entry in the matrix R. Similarly, substituting (94) in (92) one finds that the expansion coefficients x_n are given by

$$x_n(t) = e^{\lambda_n(t-t_0)} \langle Z_0, \psi_n^* \rangle_H + \sum_{j=1}^{M} \sum_{k=1}^{M_2} \int_{t_0}^{t} e^{\lambda_n(t-\tau)} \langle \beta_j, \psi_n^* \rangle_H \omega_k(\tau)d\tau\, \alpha_{jk} \qquad (98)$$

which for n=1, 2, ..., M_1 and evaluated at t_1 also define the linear constraints (93).

In order to reformulate the optimal control problem in the context of functional analysis it is convenient to introduce the following

function spaces: H_s shall represent the Hilbert space of time dependent sequences, with inner product

$$< x, y >_{H_s} = \int_{t_0}^{t_1} \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} y_j(t)\, \rho_{nj}\, x_n(t)\, dt \qquad (99)$$

where the coefficients ρ_{nj} are defined in (96). H_c will denote the finite dimensional Hilbert space of real matrices with inner product

$$<\alpha, \phi>_{H_c} = k_0 \sum_{j=1}^{M_2} \sum_{k=1}^{M_2} \sum_{n=1}^{M} \sum_{i=1}^{M} \alpha_{nj}\, \beta_{njki}\, \phi_{ik} \qquad (100)$$

where the coefficients β_{njki} are defined in (97) and the variables α_{nj} and ϕ_{ik} denote the nj-th and ik-th entries in α and ϕ respectively. Finally H_T shall represent the finite dimensional Hilbert space with inner product

$$< \xi, \gamma >_{H_T} = \sum_{i=1}^{M_1} \xi_i \gamma_i \qquad (101)$$

5.2 The Minimum Norm Formulation

With these definitions the suboptimal control problem can now be formulated in the following form: Find the control element α in H_c that minimizes

$$J(U) = ||\tilde{F}\, \alpha - \hat{y}||_{H_s} + ||\alpha||_{H_c} \qquad (102)$$

and satisfies the linear constraint

$$\xi = \tilde{T}\, \alpha \qquad (103)$$

where $\tilde{F}$ is a linear transformation from H_c into H_s with entries defined by

$$\tilde{F}_n(t)\alpha = \sum_{j=1}^{M} \sum_{k=1}^{M_2} g_{nk}(t) < \beta_j, \psi_n^* >_H \alpha_{jk} \tag{104}$$

$n=1, 2, \ldots$

where

$$g_{nk}(t) = \int_{t_0}^{t} e^{\lambda_n(t-\tau)} \omega_k(\tau)d\tau \ . \tag{105}$$

$\tilde{T}$ is a linear transformation from H_c into H_T with entries $\tilde{F}_n(t_1)$; $n=1, 2, \ldots, M_1$. ξ is an element of H_T given by

$$\xi_n = x_{1_n} - e^{\lambda_n(t_1-t_0)} < Z_0, \psi_n^* >_H \tag{106}$$

and $\hat{y}$ is a function defined in H_s by

$$\hat{y}_n(t) = z_n(t) - e^{\lambda_n(t-t_0)} <Z_0, \psi_n^* >_H \ . \tag{107}$$

From Porter's result it follows the unique solution to this problem is given by

$$\alpha = [I + \tilde{F}^{*}\tilde{F}]^{-1} [\tilde{T}^{+}\eta + \tilde{F}^{*}\hat{y}] \tag{108}$$

where η is the unique element of H_T satisfying

$$\xi = \hat{T}[I + \tilde{F}^{*}\tilde{F}]^{-1}\,[\hat{T}^{+}\eta + \tilde{F}^{*}\hat{y}] \,. \tag{109}$$

The adjoint $\tilde{F}^{*}$ is defined by the inner product relation

$$< \tilde{F}\alpha,\ y >_{H_s} = < \alpha,\ \tilde{F}^{*}y >_{H_c} \,. \tag{110}$$

Substituting (104) in the left side of (110) one finds that

$$<\tilde{F}\alpha,\ y>_{H_s} = \int_{t_0}^{t_1} \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \sum_{j=1}^{M} \sum_{k=1}^{M_2} y_i(t)\ \rho_{ni} <\beta_j,\ \psi_n^{*}>_H\ g_{nk}(t)\alpha_{jk} \tag{111}$$

and introducing a new index p which uniquely corresponds to pairs (j, k) it follows that (111) can be rewritten in the form

$$< \tilde{F}\alpha,\ y >_{H_s} = \sum_{p=1}^{M_3} \phi_p\,(y)\ \alpha_p \tag{112}$$

where $M_3 = M \times M_2$ and ϕ_p is a functional in H_s defined by

$$\phi_p(y) = \int_{t_0}^{t_1} \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} < \beta_j,\ \psi_n^{*} >_H\ g_{nk}(t)\ \rho_{ni}\ y_i(t)\ dt \,. \tag{113}$$

In view of (112), one finds by inspection that the adjoint $\tilde{F}^{*}$ is given by

$$\tilde{F}^{*}y = k_0^{-1}[\underline{\phi}(y)\ \beta^{-1}]^{T} \tag{114}$$

where β is the positive definite matrix of order M_3 with entries defined in (97). $\underline{\phi}$ is a column matrix transformation from H_s into H_c with entries

ϕ_p given in (113).

Similarly, it can be shown that the adjoint $\tilde{T}^*$ is given by

$$\tilde{T}^* \xi = k_0^{-1} \beta^{-1} \underline{\theta} \xi \tag{115}$$

where $\underline{\theta}$ is a $M_3 \times M_1$ - matrix with entries

$$\theta_{pn} = < B_j, \psi_n^* >_H g_{nk}(t_1) \tag{116}$$

in which p uniquely corresponds to the pair (j, k).

Finally, it follows from (114), (113) and (104) that the operator $[\tilde{F}^* \tilde{F}]$ is a matrix of dimension M_3 defined by

$$\tilde{F}^* \tilde{F} = k_0^{-1} \beta^{-1} \hat{F} \tag{117}$$

where $\hat{F}$ is a positive definite matrix with entries given by

$$\hat{f}_{qp} = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} < b_j, \psi_n^* >_H < B_r, \psi_i^* >_H \hat{g}_{nmik} \tag{118}$$

where

$$\hat{g}_{nmik} = \int_{t_0}^{t_1} \rho_{ni} g_{im}(t) g_{nk}(t) \, dt \tag{119}$$

and the indices q and p correspond uniquely to pairs (r, m) and (j, k) respectively.

5.3 Discussion

The reader may have noticed that H_c is in fact a finite dimensional subspace of the Hilbert space H_1, which as it may be recalled is endowed with the norm

$$||u||^2_{H_1} = k_0 \int_{t_0}^{t_1} U^T(t)\, R\, U(t)dt \,. \tag{120}$$

It follows from this observation that if the functions $\{\omega_k(t)\}$ are dense in H_1, then the optimal control in H_c is a suboptimal element of H_1 that tends to the optimal control as the number of functions M_2 in (94) tends to infinity. In order to show that this assertion is in fact true, it is convenient to point out that minimizing the performance index (8) is equivalent to minimizing

$$J(u) = ||\overline{F}\, U - \hat{y}||_{H_s} + ||U||_{H_1} \tag{121}$$

where $\overline{F}$ denotes the transformation from H_1 into H_s defined by

$$\overline{F} = COL[\overline{f}_1, \ldots, \overline{f}_n, \ldots] \tag{122}$$

and

$$\overline{f}_n(t;\tau)\, U(\tau) = \sum_{j=1}^{M} \int_{t_0}^{t} e^{\lambda_n(t-\tau)} < B_j, \psi_n^* >_H u_j(\tau)d\tau \tag{123}$$

If $\hat{U}$ denotes the optimal control function in H_1, then for any control element U in H_c it follows that

$$J(\hat{U}) \leq J(U) \tag{124}$$

Also,

$$J(U)-J(\hat{U}) = ||\bar{F}U-\hat{y}||_{H_s} - ||\bar{F}\hat{U}-\hat{y}||_{H_s} + ||U||_{H_1} - ||\hat{U}||_{H_1}$$

$$\leq ||\bar{F}(U-\hat{U})||_{H_s} + ||U-\hat{U}||_{H_1} \ . \tag{125}$$

Given that the functions ω_m are dense in H_1, for any positive real number σ, there is a positive integer M_2 such that

$$||U-\hat{U}||_{H_1} \leq \sigma \ . \tag{126}$$

Assuming that $\bar{F}$ is bounded, it follows from (125) that

$$J(U) \leq J(\hat{U}) + [||\bar{F}||+1] \ ||U-\hat{U}||_{H_1} \leq J(\hat{U}) + \varepsilon \tag{127}$$

where $||\bar{F}||$ denotes the norm of $\bar{F}$, and

$$\varepsilon = [||\bar{F}||+1] \ \sigma \ . \tag{128}$$

If U is the element of H_c which minimizes (121), then

$$J(\hat{U}) \leq J(U) \leq J(\hat{U}) + \varepsilon \ , \tag{129}$$

and since ε can be made arbitrarily small, it follows that J(U) tends

to $J(\hat{U})$ as M_2 tends to infinity.

5.4 An Example

Consider again the linear, homogeneous slab reactor model of section 4.3 of Chapter III,

$$\frac{\partial \psi}{\partial t}(r, t) = VD \frac{\partial^2}{\partial r^2} \psi(r, t) + V[v\Sigma_f - \Sigma_a] \psi(r, t)$$

$$- V \sum_{i=1}^{M} u_i(t) \phi_0(r) \delta(r - r_i) \qquad (130)$$

with boundary condition

$$\psi(0, t) = \psi(b, t) = 0 \qquad (131)$$

and the steady state distribution given by

$$\phi_0(r) = \phi_M \sqrt{\frac{2}{b}} \sin\left(\frac{\pi r}{b}\right) . \qquad (132)$$

In this particular case the operator $VD \frac{\partial^2}{\partial r^2} + V[v\Sigma_f - \Sigma_a]$ is self-adjoint and generates the orthonormal eigenfunctions

$$\psi_n(r) = \psi_n^*(r) = \sqrt{\frac{2}{b}} \sin\left(\frac{n\pi}{b} r\right) \qquad (133)$$

and eigenvalues

$$\lambda_n = \left[\frac{(v\Sigma_f - \Sigma_a)}{\pi^2 D} b^2 - n^2\right] V \frac{\pi^2}{b^2} D \qquad (134)$$

Also, for simplicity, Q and R are assumed to be the identity matrices. The desired trajectory $Z(r, t)$ and the desired final distribution $Z_1(r)$ are both chosen to be zero.

$$Z(r, t) = Z_1(r) = 0. \tag{135}$$

If the functions $\{\omega_j(t)\}$ are chosen to be the unit pulses

$$\omega_j(t) = \begin{cases} 1 & , \quad \tau_{j-1} < t \leq \tau_j \\ 0 & , \quad \text{elsewhere} \end{cases} \tag{136}$$

where

$$\tau_j = \frac{(t_1 - t_0)}{M_2} j \qquad j=1, 2, \ldots, M_2 \tag{137}$$

then the coefficients of the modal expansion (92) satisfy

$$x_n(t) = e^{\lambda_n (t-t_0)} \langle Z_0, \psi_n \rangle_H + \sum_{j=1}^{M} \sum_{k=1}^{M_2} g_{nk}(t) \langle B_j, \psi_n \rangle_H \alpha_{jk} \tag{138}$$

where

$$\langle B_j, \psi_n \rangle = - \sqrt{\frac{2}{b}} \phi_0(r_j) V \sin\left(\frac{n\pi}{b} r_j\right) \tag{139}$$

and

$$g_{nk}(t) = \int_{t_0}^{t} e^{\lambda_n(t-\tau)} \omega_k(\tau) d\tau \qquad (140)$$

which for $\lambda_n = 0$ takes on the values:

$$g_{nk}(t) = \begin{cases} 0 & , \quad t \leq \tau_{k-1} \\ t - \tau_{k-1} & , \quad \tau_{k-1} < t \leq \tau_k \\ \dfrac{(t_1 - t_0)}{M_2} & , \quad \tau_k < t \end{cases} \qquad (141)$$

and for $\lambda_n \neq 0$,

$$g_{nk}(t) = \begin{cases} 0 & , \quad t \leq \tau_{k-1} \\ \lambda_n^{-1}[e^{\lambda_n(t-\tau_{k-1})} - 1] & , \quad \tau_{k-1} < t \leq \tau_k \\ \lambda_n^{-1}[e^{\lambda_n(t-\tau_{k-1})} - e^{\lambda_n(t-\tau_k)}], & \quad \tau_k < t \end{cases} \qquad (142)$$

The adjoint $\tilde{F}^*$ is defined by

$$F^* y = k_0^{-1} \underline{\phi}(y) \qquad (143)$$

where

$$\phi_p(y) = - \sum_{n=1} \langle B_j, \psi_n \rangle_H \int_{t_0}^{t_1} g_{nk}(t) e^{\lambda_n(t-t_0)} dt \, \langle Z_0, \psi_n \rangle_H, \qquad (144)$$

and the index p is given by

$$p = M_2(j-1) + k \,. \tag{145}$$

Also, the matrix $\tilde{\tilde{F}}^{*}\tilde{\tilde{F}}$ is defined by

$$\tilde{\tilde{F}}^{*}\tilde{\tilde{F}} = k_0^{-1}\hat{F} \tag{146}$$

where $\hat{F}$ denotes a matrix with entries

$$\hat{f}_{pq} = \sum_{n=1}^{\Gamma} \langle B_i, \psi_n \rangle_H \langle B_j, \psi_n \rangle_H \hat{g}_{nmnk} \tag{147}$$

and

$$\hat{g}_{nmnk} = \int_{t_0}^{t_1} g_{nm}(t)\, g_{nk}(t)\, dt. \tag{148}$$

The indices p and q in (147) are determined uniquely from

$$q = M_2(j-1) + m \tag{149}$$

$$p = M_2(i-1) + k \tag{150}$$

where m and k take on integer values in $[1, M_2]$ and j and i take on integer values in $[1, M]$.

The adjoint $\tilde{T}^{*}$ is given by

$$\tilde{T}^{*} = k_0^{-1}\,\theta \tag{151}$$

where θ is a rectangular matrix with entries

$$\theta_{pn} = \langle B_j, \psi_n \rangle_H \; g_{nk}(t_1) \; . \tag{152}$$

Also,

$$\xi_n = - e^{\lambda_n (t_1 - t_0)} \langle Z_0, \psi_n \rangle_H \; . \tag{153}$$

6. Numerical Results

The slab reactor model considered in the following numerical examples is based upon the data given by Wiberg in reference ([11], p. 359). The complete set of data used in these examples is presented in Table 1.

In this particular case, the eigenvalues are real. The first eigenvalue is close to zero and all the other eigenvalues are negative.

For the case where no control action is applied, Figure 1 shows the trajectory of the flux deviations along the time-interval [0.0, 2.5] seconds. At the end of the time-interval all the higher harmonics have decayed almost completely and only the fundamental mode remains virtually unchanged.

The flux deviations are normalized by the maximum value ϕ_M.

In all the cases, the control functions considered belong to the finite dimensional space spanned by pulse-functions of duration equal to 0.25 seconds.

Case 1.

Figure 2 shows the trajectory of the flux deviations for the case where the control action minimizes the performance index.

$$J(U) = \int_{t_0}^{t_1} \int_{0}^{b} \psi^2(r, t)\, dr dt + k_0 \int_{t_0}^{t_1} \sum_{i=1}^{2} u_i(t)\, dt \qquad (154)$$

and in addition satisfies the constraint that the fundamental and the first two harmonics should vanish at the end of the time-interval.

The optimal control is given in Table 2.

Migration Length	$D/\Sigma_a = 160\ cm^2$
Average Neutron Lifetime	$[\Sigma_a V]^{-1} = 0.1$ sec
Infinite Homogeneous Multiplication Constant	$v\Sigma_f/\Sigma_a = 1.0256$
Neutrons/Fission	$v = 2.5$
Diffusion Coefficient	$D = 0.5070$ cm
Reactor Width	$b = 250$ cm
Time Interval $[t_0, t_1]$	$t_0 = 0.0$ sec
	$t_1 = 2.5$ sec
Weighting Parameter	$k_0 = 10.000$
Number of Rods	$M = 2$
Number of Linear Constraints in Equation (153)	$M_1 = 3$
Duration of Pulses $\omega_k(t)$	$[t_1-t_0]/M_2 = 0.25$ sec
Control Rod Location	$r_1 = 66$ cm, $r_2 = 150$ cm
Initial State Distribution	$Z_0(r) = \phi_M[0.05\psi_1(r)+0.03\psi_2(r)+0.01\psi_3(r)]$
Desired State Distribution	$Z(r) = 0.0$

Table 1 Data For Numerical Examples

Figure 1 The Uncontrolled Flux Distribution

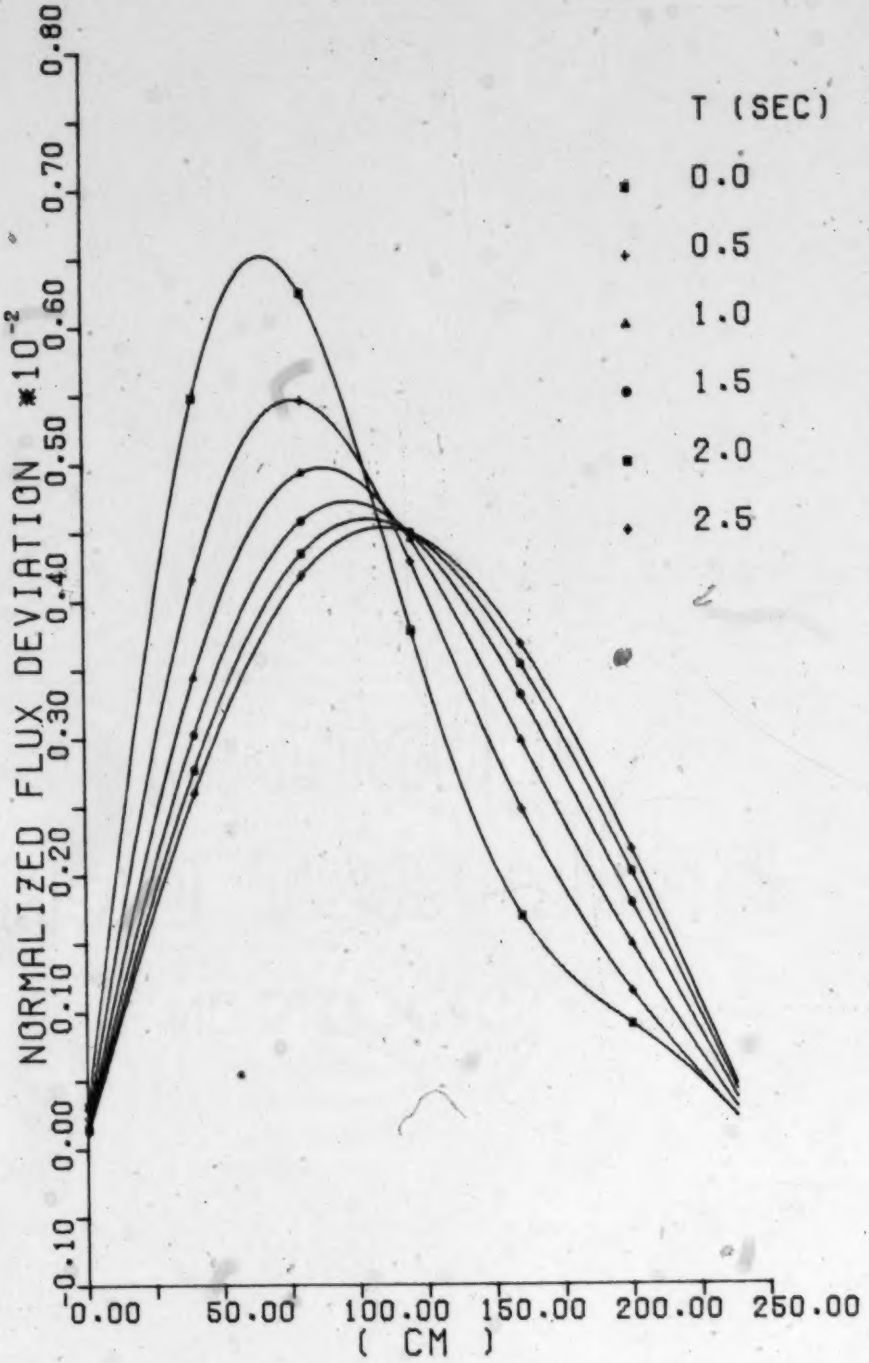
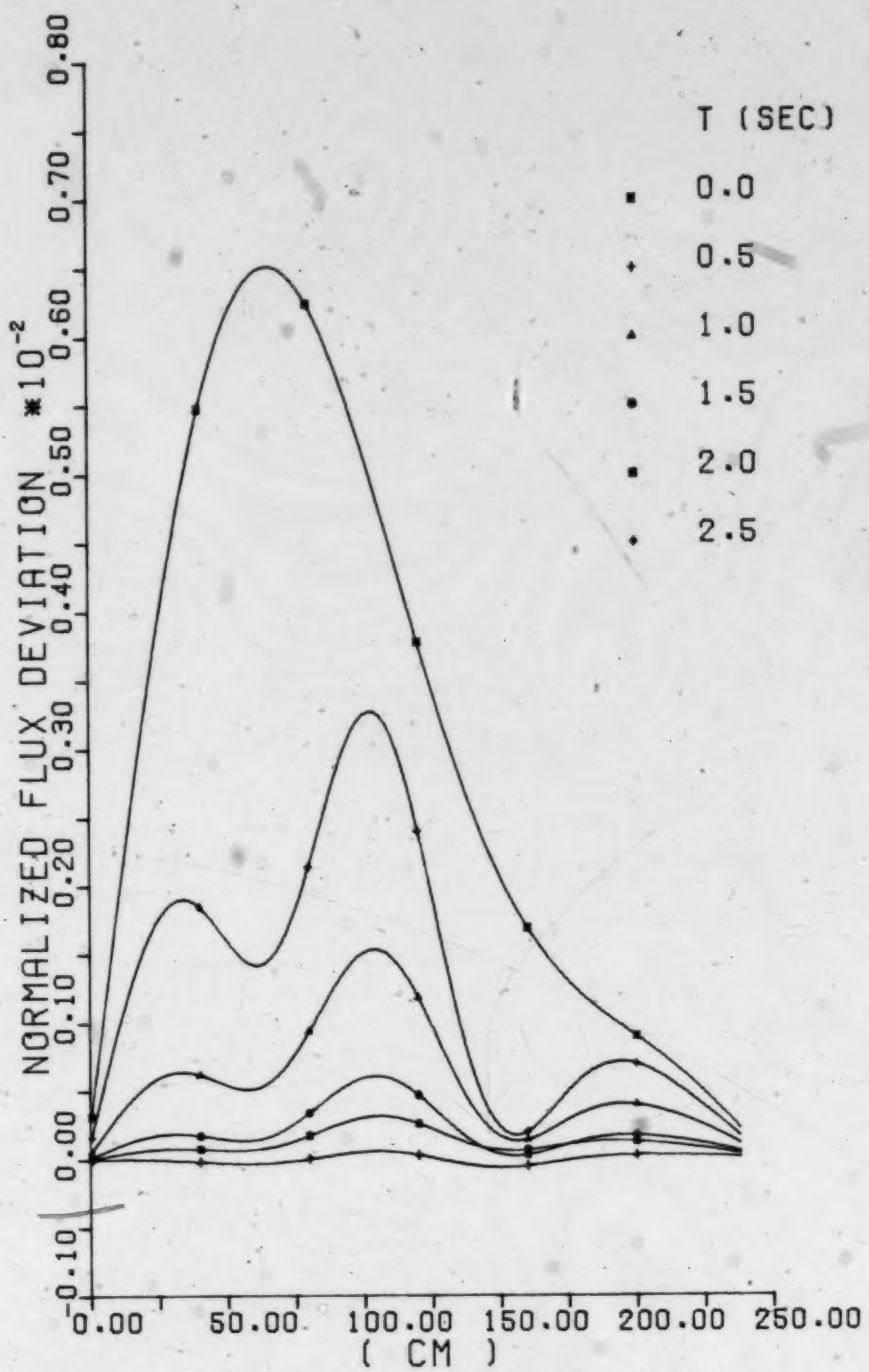

NORMALIZED FLUX DEVIATION *10⁻²
-0.10
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.00
50.00
100.00
150.00
200.00
250.00
(CM)
T (SEC)
0.0
0.5
1.0
1.5
2.0
2.5

Figure 2 The Controlled Flux Distribution:

Case 1.


T (SEC)
0.0
0.5
1.0
1.5
2.0
2.5
NORMALIZED FLUX DEVIATION *10⁻²
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
-0.10
0.00
50.00
100.00
150.00
200.00
250.00
(CM)

t	Case 1		Case 2	
	rod at r=66cm	rod at r=150cm	rod at r=66cm	rod at r=150cm
(sec)	$u_1(t)$	$u_2(t)$	$u_1(t)$	$u_2(t)$
0.00	0.274×10^{-2}	0.139×10^{-2}	0.274×10^{-2}	0.139×10^{-2}
0.25	0.168×10^{-2}	0.915×10^{-3}	0.168×10^{-2}	0.918×10^{-3}
0.50	0.105×10^{-2}	0.618×10^{-3}	0.105×10^{-2}	0.621×10^{-3}
0.75	0.673×10^{-3}	0.426×10^{-3}	0.664×10^{-3}	0.420×10^{-3}
1.00	0.432×10^{-3}	0.289×10^{-3}	0.420×10^{-3}	0.282×10^{-3}
1.25	0.271×10^{-3}	0.189×10^{-3}	0.265×10^{-3}	0.188×10^{-3}
1.50	0.176×10^{-3}	0.127×10^{-3}	0.165×10^{-3}	0.122×10^{-3}
1.75	0.124×10^{-3}	0.948×10^{-4}	0.982×10^{-4}	0.761×10^{-4}
2.00	0.850×10^{-4}	0.714×10^{-4}	0.520×10^{-4}	0.414×10^{-4}
2.25	0.505×10^{-4}	0.569×10^{-4}	0.162×10^{-4}	0.132×10^{-4}

Table 2 Optimal Control Functions for Cases 1 and 2

Case 2.

Figure 3 shows the trajectory of the flux deviations for the case where the control function is only required to minimize the cost functional (154). Although the results of Figures 2 and 3 are almost identical, it is clearly evident from Figure 3 that at the end of the time-interval the fundamental mode is still present.

The optimal control for this case is given in Table 2.

Case 3.

Figure 4 shows the trajectory of the flux deviation for the case where the control is required to satisfy the constraint that the fundamental and the two first harmonics should vanish at the end of the time-interval and, in addition, should minimize the performance index

$$J(U) = \int_{t_0}^{t_1} \sum_{i=1}^{2} u_i(t) . \qquad (155)$$

The effect of the optimal control function upon the trajectory compares poorly against the first two cases, as should be expected. The optimal control function computed in this case constitutes the effortless way of satisfying the given constraint on the final state. The control function is given in Table 3.

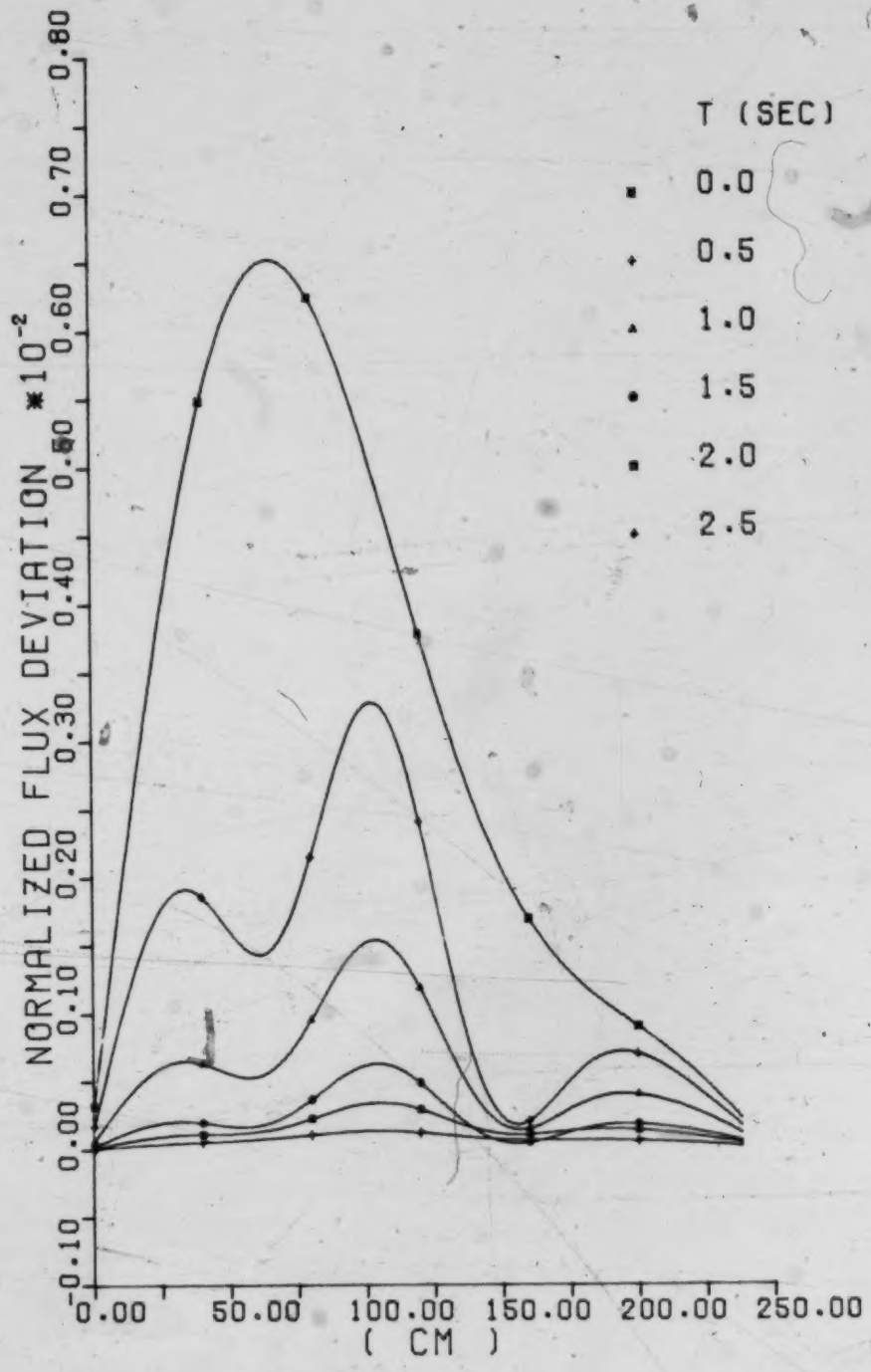
Case 4.

Figure 5 shows the flux deviations for the case where the control function minimizes the performance index

$$J(U) = \int_{o}^{b} \psi^2(r, t_1)dr + k_0 \int_{t_0}^{t_1} \sum_{i=1}^{2} u_i(t)dt. \qquad (156)$$

Figure 3 The Controlled Flux Distribution:

Case 2.


T (SEC)
0.0
0.5
1.0
1.5
2.0
2.5
NORMALIZED FLUX DEVIATION *10⁻²
-0.10
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.00
50.00
100.00
150.00
200.00
250.00
(CM)

t	Case 3		Case 4	
	rod at r=66cm	rod at r=150cm	rod at r=66cm	rod at r=150cm
sec	$u_1(t)$	$u_2(t)$	$u_1(t)$	$u_2(t)$
0.00	0.593×10^{-3}	0.578×10^{-3}	0.564×10^{-3}	0.183×10^{-4}
0.25	0.497×10^{-3}	0.481×10^{-3}	0.118×10^{-2}	0.663×10^{-3}
0.50	0.500×10^{-3}	0.478×10^{-3}	0.115×10^{-2}	0.676×10^{-3}
0.75	0.600×10^{-3}	0.574×10^{-3}	0.482×10^{-3}	0.551×10^{-4}
1.00	0.606×10^{-3}	0.571×10^{-3}	0.443×10^{-3}	0.724×10^{-3}
1.25	0.517×10^{-3}	0.469×10^{-3}	0.103×10^{-2}	0.728×10^{-3}
1.50	0.526×10^{-3}	0.463×10^{-3}	0.975×10^{-3}	0.749×10^{-3}
1.75	0.640×10^{-3}	0.550×10^{-3}	0.270×10^{-3}	0.133×10^{-3}
2.00	0.662×10^{-3}	0.535×10^{-3}	0.173×10^{-3}	0.132×10^{-3}
2.25	0.600×10^{-3}	0.414×10^{-3}	0.578×10^{-3}	0.641×10^{-3}

Table 3 Optimal Control Functions For Cases 3 and 4.

Figure 4 The Controlled Flux Distribution:

Case 3.

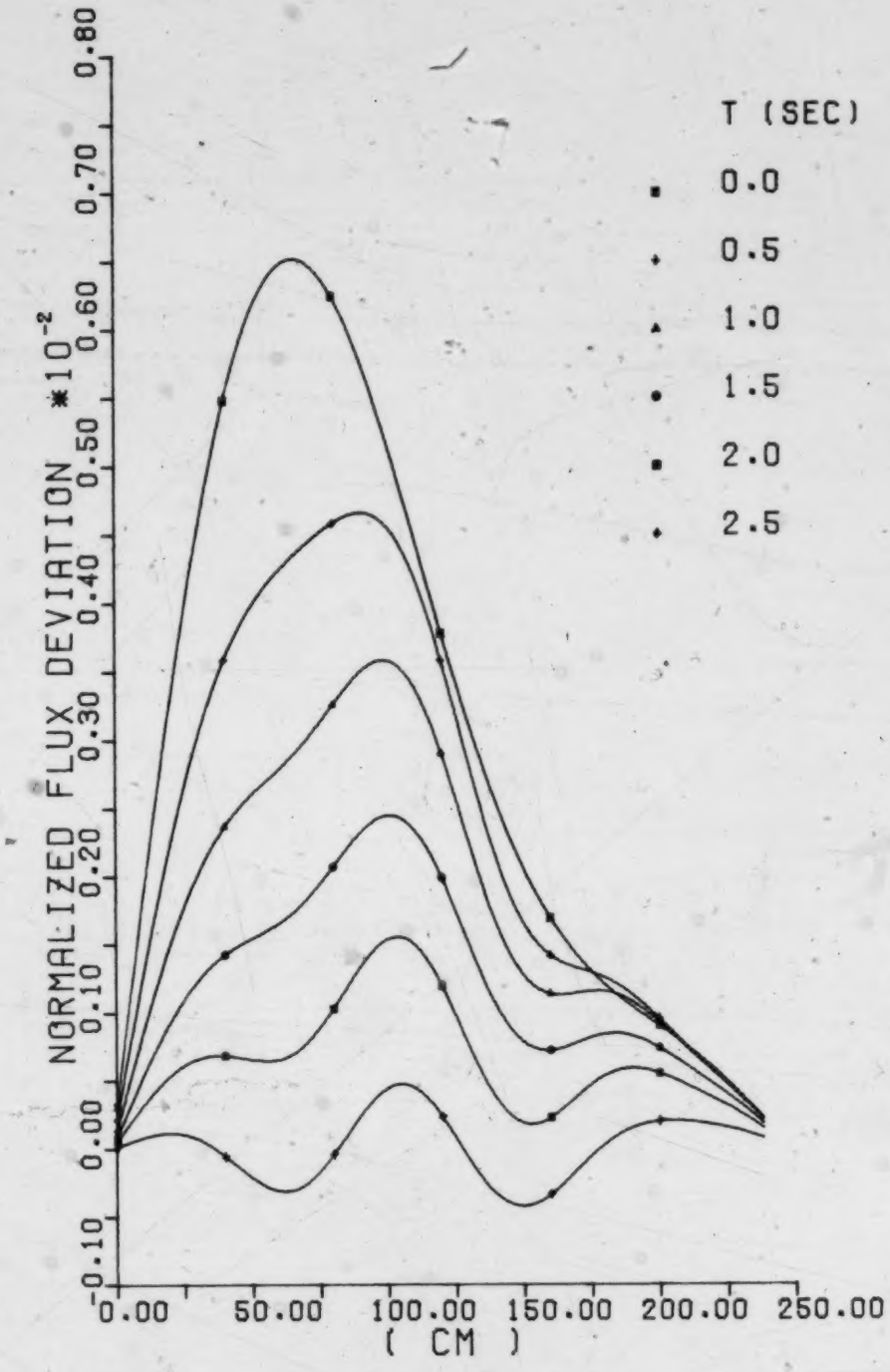
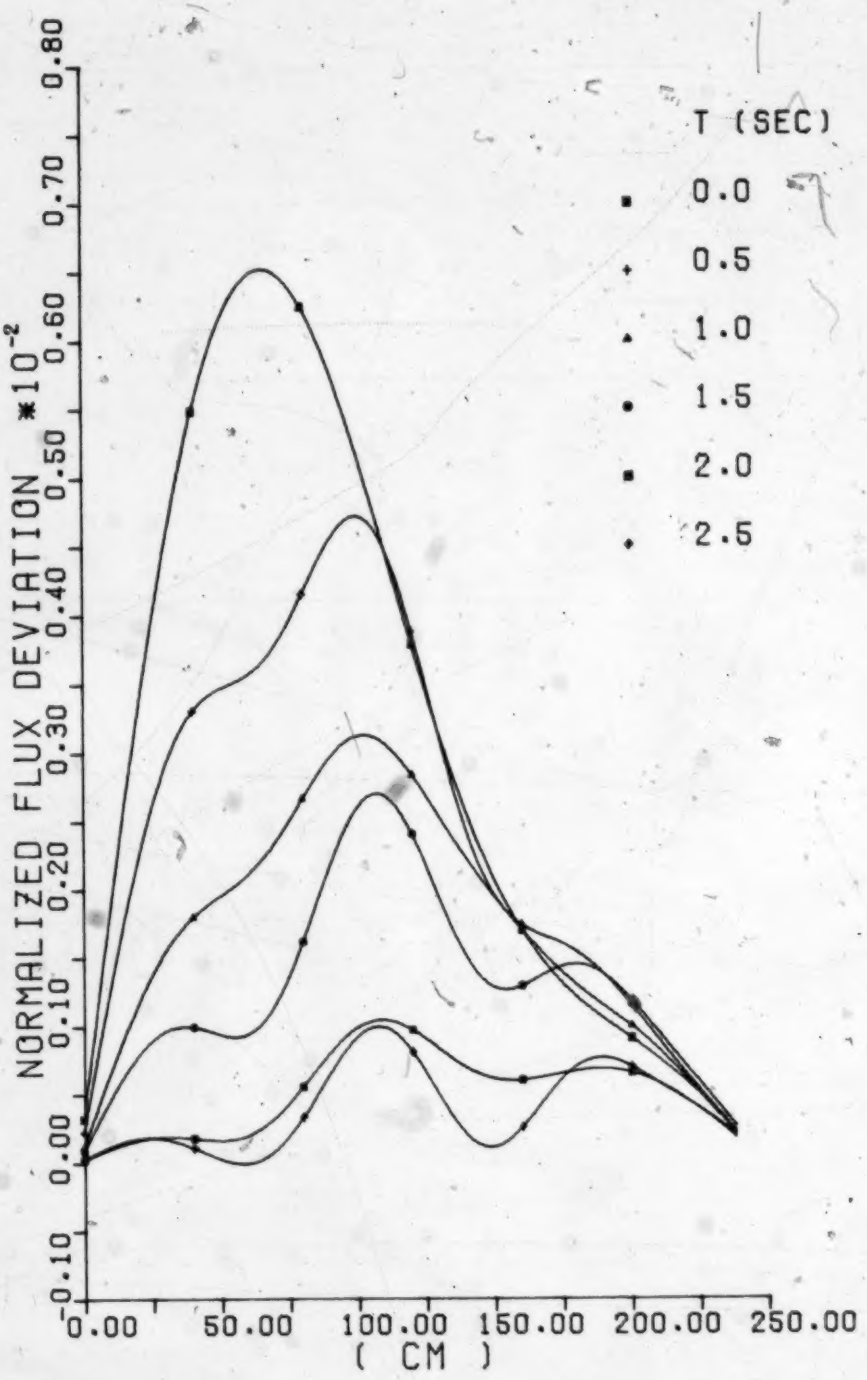

T (SEC)
0.0
0.5
1.0
1.5
2.0
2.5
NORMALIZED FLUX DEVIATION *10⁻²
-0.10
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.00
50.00
100.00
150.00
200.00
250.00
(CM)

Figure 5 The Controlled Flux Distribution:

Case 4.


T (SEC)
0.0
0.5
1.0
1.5
2.0
2.5
NORMALIZED FLUX DEVIATION *10^-2
-0.10
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.00
50.00
100.00
150.00
200.00
250.00
(CM)

The performance of the optimal control in this case does not compare favorably against the first two cases where the deviations of the flux are penalized along the trajectory. Although this result should be expected, it should be mentioned also that the comparison of the present against the first two cases is not entirely valid, since for the same weighting parameter k_0 in the performance indices (154) and (156) the absence of the time-integral in the first term of (156) implicitly gives more weight to the second term of the performance index, which of course penalizes the control effort. Consequently, the norm of the optimal control in this case is almost two times smaller than the corresponding norms in the first two cases. The optimal control is given in Table 3.

The computations were performed on the Amdahl 470/6 digital computer of the University of Alberta.

CHAPTER V

SYMMETRY REDUCTION OF REACTOR SYSTEMS AND ITS APPLICATION TO OPTIMAL CONTROL

1. Introduction

Although the discussion in the previous chapters concerning the application of the minimum norm techniques applies to linear distributed parameter models in general, the practical implementation of the derived theoretical results to more realistic reactor models, involving for example geometrical configurations in 2 or 3 dimensions and including the dynamics of fission products such as xenon and iodine, can only be made possible if efficient methods for treating both the mathematical models and the resulting conditions for optimality are utilised.

One of the main difficulties that arise in the case of realistic 3-dimensional reactor models is posed by the large number of control devices involved. As the reader may recall from the discussion in the previous chapters, the conditions for optimality derived for a variety of control problems include at least as many equations as there are control devices in the core. It is not uncommon for large reactors to have 14 or more of such reactivity control devices [96].

This chapter is devoted to the problem of decoupling, or reducing, either the mathematical reactor models or the derived conditions for optimality into more tractable and independent submodels of lower order.

The reduction approach presented here takes advantage of the geometrical symmetry that may exist in a nuclear reactor configuration.

Its foundations are based on the well-developed theory of the representations of abstract groups [34, 94, 95].

2. Some Basic Concepts of Group Theory and Symmetry Principles

The discussion of this section is aimed at establishing the operational notation that will be utilized in the chapter. Only the basic concepts of group theory that are needed to describe the approach to reducing linear distributed parameter systems by means of symmetry principles are cited. The detailed discussion of definitions and group theoretic results is left to the references [34, 94, 95].

An abstract group is a set of distinct elements with a binary multiplication law such that

a) Multiplication is defined for every ordered pair of elements of the group, is closed, single-valued and associative.

b) There exists an identity element I in the group.

c) Every element has an inverse relative to the identity I.

The number of elements in a group is called the order of the group.

A representation, Γ, of a group, g, is a group in which the elements are matrices. The matrices in Γ are associated with the elements of the group g through a many-to-one correspondence which preserves multiplication.

Two sets of square matrices, say $\{D_i\}$ and $\{D_i'\}$ for i=1, 2, ..., h are said to be equivalent if there exists a square matrix M, such that

$$D_i' = M^{-1} D_i M \quad i=1, 2, \ldots, h \tag{1}$$

for some ordering of the matrices $\{D_i'\}$.

A representation of a group is said to be reducible if it is

equivalent to a representation of g composed of matrices in block-diagonal form.

The character of a representation with matrices $\{D_i\}$ is the set of quantities $\{\chi_i\}$ where χ_i is the trace of the matrix D_i.

If D_1 and D_2 are elements of a group then the element $D_1^{-1} D_2 D_1$ is said to be the conjugate of D_2 with respect to D_1.

A class of a group is a maximal set of mutually conjugate elements.

The main theorem of the representation of groups states that every group has exactly as many inequivalent irreducible representations as there are classes in the group. Furthermore, if $\Gamma^{(q)} = \{D_j^{(q)}\}$ and $\Gamma^{(p)} = \{D_j^{(p)}\}$; j=1, 2, ..., h are any two of these representations, then the matrix entries satisfy the following orthogonality relations:

$$\text{a)} \quad \sum_{j=1}^{h} D^{(q)}_{j_{mn}} \; [D_j^{(p)}]^{-1}_{nm} = 0 \quad \text{if } q \neq p \tag{2}$$

$$\text{b)} \quad \sum_{j=1}^{h} D^{(q)}_{j_{mn}} \; [D_j^{(q)}]^{-1}_{ik} = 0 \quad \text{if } n \neq i \tag{3}$$

and/or $m \neq k$.

$$\text{c)} \quad \sum_{j=1}^{h} D^{(q)}_{j_{mn}} \; [D_j^{(q)}]^{-1}_{nm} = \frac{h}{\ell_q} \tag{4}$$

where $D^{(q)}_{j_{mn}}$ denotes the mn-th entry in the jth matrix of the representation $\Gamma^{(q)}$. $[D_j^{(q)}]^{-1}$ denotes the inverse of $D_j^{(q)}$. h and ℓ_q represent the order of the group and the dimension of each of the matrices in the representation $\Gamma^{(q)}$, respectively.

The symmetry of a body is described by giving the set of all those transformations which preserve distances and bring the body into coincidence with itself. Any such transformation is called a symmetry transformation.

For a given body a complete list of symmetry transformations satisfy the group properties. This is called the symmetry group of the body.

If a body is finite in extent, the symmetry transformations can all be built up from two fundamental types:

a) Rotation through a definite angle about some axis.

b) Mirror reflection in a plane.

In what follows, the ordered collection $C = \{C_i\}$ will represent the set of reactor regions, the union of which constitute the reactor core. The regions are disjoint and the only common points between any two contiguous regions are located at their shared boundaries.

A symmetry transformation defined on the core, which physically represents a positional interchange of reactor regions such that distances between points of the core are preserved, can be seen to represent in the context of abstract sets a permutation operation defined on the ordered set C.

If R denotes a symmetry transformation on the core then $RC = \{RC_i\}$ denotes the image of the set C under R, and it is composed of the same collection of regions but now in different order.

For example, the rectangle of Figure 6, which arbitrarily has been decomposed into four equivalent disjoint regions, has its symmetry described by the following symmetry transformations:

a) σ_1, reflection with respect to the y-axis.

b) σ_2, reflection with respect to the x-axis

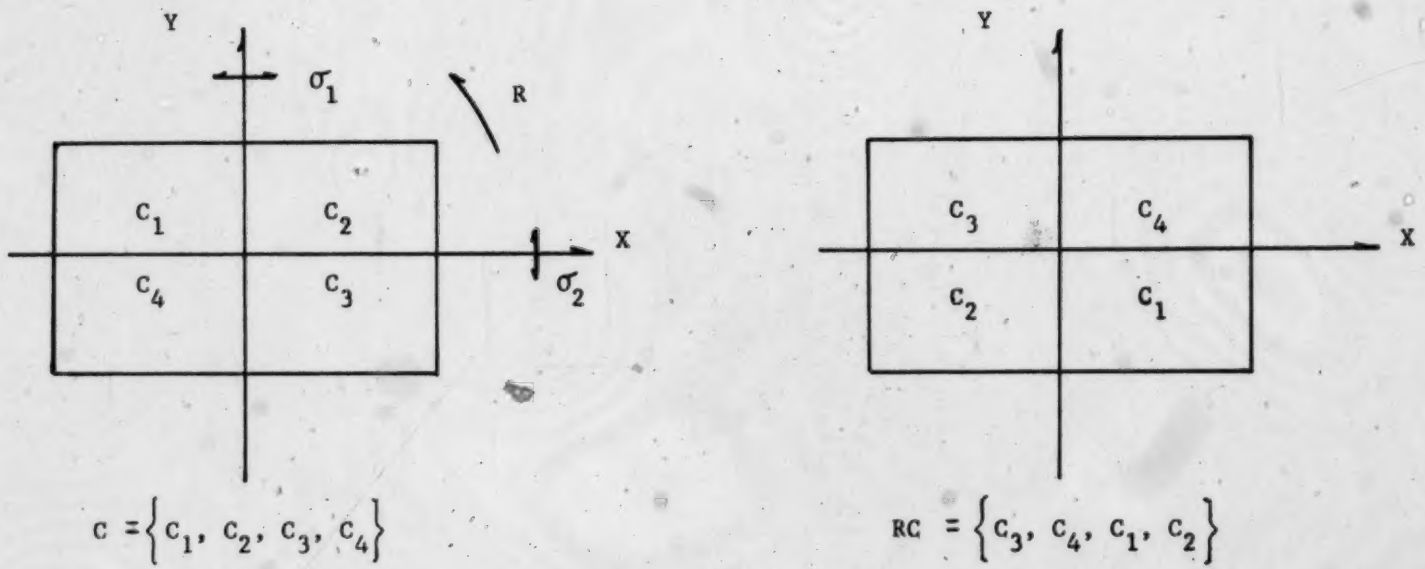


Figure 6 Symmetry Transformations on a Rectangle.

c) R, rotation of π radians with respect to the center.

d) I, the identity.

Applying the transformation R upon the rectangle, one finds that the regions 1 and 3 are interchanged, and so are the regions 4 and 2.

Therefore, making use of our notation, it follows that,

$$RC_1 = C_3$$
$$RC_2 = C_4$$
$$RC_3 = C_1$$
$$RC_4 = C_2$$

and the ordered set $C = \{C_1, C_2, C_3, C_4\}$ is transformed into $RC = \{C_3, C_4, C_1, C_2\}$.

The dependency of a function $\psi(r)$ on the spatial variable r, which is defined on the region occupied by the core C, will be explicitly shown in the form $\psi(C)$. This notation is particularly convenient since it facilitates the extension of the concept of a symmetry transformation on a body, or on an ordered set C, to the concept of a symmetry transformation on functions of the spatial variable r. In this manner, if R is a symmetry transformation belonging to the symmetry group of the body C, then by definition, the image $R\,\psi(C)$ of a function $\psi(C)$ under R is given by the function ψ defined over the transformed body, RC. This suggests the notation

$$R\,\psi(C) \triangleq \psi(RC) \,. \qquad (5)$$

If $\psi(C)$ is an element of some linear space, then it is clear that R defines a linear transformation from the same linear space into itself.

It is pertinent to mention here that the function defined on the reactor core C could either be defined on a continuum, as is the case for the flux distribution, or defined at discrete locations, as is the case for the control functions.

For example, the distribution $\psi(C)$ defined on the rectangle of Figure 6 by

$$\psi(x, y) = \cos(\frac{\pi x}{a}) \sin(\frac{2\pi y}{b}) , \tag{6}$$

where x and y take on values over [-a/2, a/2] and [-b/2, b/2] respectively, is transformed by the reflection operation σ_2 into

$$\sigma_2 \psi(x, y) = -\cos(\frac{\pi x}{a}) \sin(\frac{2\pi y}{b}) , \tag{7}$$

which in our compact notation becomes

$$\sigma_2 \psi(C) = -\psi(C) . \tag{8}$$

Similarly, if u_i denotes some specific property associated with the region C_i, in the rectangle of Figure 6, then the vector $U(C) = COL[u_1, u_2, u_3, u_4]$ would be transformed by σ_2 as follows

$$\sigma_2 U(C) = COL[u_4, u_3, u_2, u_1] . \tag{9}$$

A collection of functions $\{f_j(C)\}$, finite or infinite, is said to

to generate a representation $\{D_i\}$ of a symmetry group $g = \{R_i\}$, associated with a body C, if for every symmetry transformation R_i and every function $f_j(C)$ the term $R_i f_j(C)$ satisfies a relation of the form

$$R_i f_j(C) = \sum_k D_{i_{jk}} f_k(C) . \qquad (10)$$

After this lengthy introduction of a new notation we are now in a position to profit from the results of the main theorem of the representation theory of groups.

Consider a Hilbert space H of functions defined on a region or body C which has a symmetry group $g = \{R_i\}$ of order h.

A representation of g can be generated by applying all the symmetry transformations R_i to any function f(C) in H. For example, if f_i denotes the function $f(R_i C)$, then the set $\{f_i\}$ i=1, 2, ..., h defines a collection of functions that also belong to H. Since for any transformation R_i in the group g and any function f_j the product $R_i f_j$ is also in the set $\{f_i\}$ it follows that the product $R_i f_j$ can be represented in the form

$$R_j f_i = \sum_{k=1}^{\ell} D_{j_{ik}} f_k , \qquad (11)$$

where ℓ is the number of unrepeated functions in the set $\{f_i\}$ and D_j denotes the j-th matrix in the representation of g generated by the function f.

In general, the representation of g generated by an arbitrary function is reducible.

It follows from the main theorem of the representation theory of

groups that an arbitrary function f(C) can be decomposed into orthogonal components, $f_n^{(q)}(C)$, which have the property that generate only the nth-row of the q-th irreducible representation of the group g. That is, if N_C denotes the number of classes in the group g, and ℓ_q defines the dimension of the matrices in the q-th irreducible representation of g, denoted by $\{\Gamma_j^{(q)}\}$ j=1, 2, ..., h. Then the function f(C) can be rewritten in the form

$$f(C) = \sum_{n=1}^{\ell_q} \sum_{q=1}^{N_C} f_n^{(q)}(C) , \tag{12}$$

where the components $f_n^{(q)}(C)$ transform under the symmetry operation R_i according to the form

$$R_i f_n^{(q)}(C) = \sum_{k=1}^{\ell_q} \Gamma_{i_{nk}}^{(q)} f_k^{(q)}(C) , \tag{13}$$

in which $\Gamma_{i_{nk}}^{(q)}$ denotes the nk-th entry in the i-th matrix of the q-th irreducible representation of g.

Furthermore, the components $\{f_n^{(q)}(C)\}$ can be obtained by applying the projection operators

$$P_n^{(q)} = \frac{\ell_q}{h} \sum_{i=1}^{h} [\Gamma_{i_{nn}}^{(q)}]^* R_i , \tag{14}$$

where n and q can take on integer values in $[1, \ell_q]$ and $[1, N_C]$ respectively, and the symbol $[\ ]^*$ denotes complex conjugate.

In summary, the projection operators $\{P_n^{(q)}\}$ decompose the space H into the direct sum of subspaces $H_n^{(q)}$,

$$H = \sum_{n=1}^{\ell_q} \sum_{q=1}^{N_c} H_n^{(q)} \qquad (15)$$

where the functions in $H_n^{(q)}$ generate the nth-row of the qth-irreducible representation of g.

3. Symmetry Reduction of Nuclear Reactor Models

The application of the group theoretic results of the previous section can drastically simplify the analysis and treatment of problems involving linear distributed parameter models which, due to the nature of the physical system that these represent, exhibit some degree of symmetry.

Consider for example the general linearized distributed reactor model of Chapter III,

$$\frac{\partial \psi}{\partial t}(r, t) = A(r)\,\psi(r, t) + B(r)\,U(t)\,, \tag{16}$$

where r is the spatial variable defined on the reactor core C. ψ denotes the state of the reactor core. For fixed r and t, $\psi(r, t)$ represents an N-dimensional vector. A(r) is a matrix spatial operator involving operators of the diffusion type. For fixed time t, U(t) is a M-dimensional vector representing the effect of the control rods. B(r) is a rectangular matrix spatial operator of the appropriate dimension.

Also, associated with the equation (16) are boundary conditions of the form

$$\psi(r, t) = 0 \tag{17}$$

at the boundary of the core, and the initial condition

$$\psi(r, t_0) = Z_0(r), \tag{18}$$

at time t_0.

In terms of our new notation the equation (16) and conditions (17) and (18) are given by

$$\frac{\partial}{\partial t}\psi(c,t) = A(c)\,\psi(c,t) + B(c)\,U(c,t), \tag{19}$$

$$\psi(c,t) = 0 \text{ at the boundary}, \tag{20}$$

and

$$\psi(c,t_0) = Z_0(c). \tag{21}$$

It should be noticed that although the vector U does not depend explicitly on the spatial variable r, clearly, it represents a function of the core since every entry in the vector U corresponds to a control rod, which in turn is associated with a certain location in the core.

If the geometrical configuration of the reactor has the symmetry group $g = \{R_i\}$ and if the reactor parameters that appear in the mathematical model are distributed over the core according to the same symmetry described by g, then it follows that the operators A and B are invariant to all the symmetry transformations R_i.

In view of the linearity of the operators A and B it is clear that in this case the application of the projection operators $\{p_n^{(q)}\}$ to equation (19) and conditions (20) and (21) yields

$$\frac{\partial}{\partial t}\psi_n^{(q)}(c,t) = A(c)\,\psi_n^{(q)}(c,t) + B(c)\,U_n^{(q)}(c,t), \tag{22}$$

$$\psi_n^{(q)}(c, t) = 0 \quad \text{at the boundary} \tag{23}$$

and

$$\psi_n^{(q)}(c, t) = Z_{0_n}^{(q)}(c) , \tag{24}$$

where

$$\psi(c, t) = \sum_{q=1}^{N_c} \sum_{n=1}^{\ell_q} \psi_n^{(q)}(c, t) \tag{25}$$

and

$$U(c, t) = \sum_{q=1}^{N_c} \sum_{n=1}^{\ell_q} U_n^{(q)}(c, t) . \tag{26}$$

Thus, the projection operators $\{p_n^{(q)}\}$ decompose the space H of state distributions and the space E of control vectors into the invariant manifolds $H_n^{(q)}$ and $E_n^{(q)}$ of A(c) and B(c) respectively. It follows from this observation that optimal control problems involving a performance index of the type encountered in Chapters III and IV,

$$J(U) = \int_{t_0}^{t_1} [||\psi-Z||_H + k_0||U||_E]dt \tag{27}$$

and mathematical models such as equation (16) can be reduced into a number of subproblems which consist of minimizing

$$J_n^{(q)}(U) = \int_{t_0}^{t_1} [||\psi-Z||_{H_n^{(q)}} + k_0||U||_{E_n^{(q)}}]dt \tag{28}$$

subject to (22), (23) and (24).

The benefit of the reductive approach is clear. While the original problem implies the search for M functions of time, $\{u_i(t)\}$, in the case of the subproblems it is the dimension of the subspaces $E_n^{(q)}$ what dictates the number of functions of time that ought to be computed. Depending on the degree of symmetry exhibited by the reactor core, these subspaces could have a dimension substantially lower than M.

It should be pointed out that the parameters appearing in the defining relation (14), for the projections $P_n^{(q)}$, can all be obtained from the properties of the symmetry group g.

For most of the symmetry groups of interest in the physical sciences these properties have been tabulated and are available in the literature. (See for example Reference [14], Appendix III).

4. An Example: Decoupling of the Optimality Conditions

The present example illustrates the benefits of using symmetry principles in the treatment of the optimality conditions which result from application of optimization techniques such as the minimum norm formulation.

As the reader may recall from example 4.3 of Chapter III, the necessary and sufficient conditions for optimality that specify the solution to the problem of minimizing the performance index

$$J(U) = \int_{t_0}^{t_1} \int_{o}^{b} \psi^2(r, t)drdt + k_0 \int_{t_0}^{t_1} \sum_{i=1}^{M} u_i^2(t)dt, \quad (29)$$

subject to the slab reactor model

$$\frac{\partial \psi(r, t)}{\partial t} = VD \frac{\partial^2}{\partial r^2} \psi(r, t) + V[v\Sigma_f - \Sigma_a]\psi(r, t)$$

$$- V \sum_{i=1}^{M} u_i(t)\, \phi_0(r)\, \delta(r-r_i)\,, \quad (30)$$

$$\phi_0(r) = \phi_M \sqrt{\frac{2}{b}}\, \sin\left(\frac{\pi r}{b}\right)\,, \quad (31)$$

with boundary conditions

$$\psi(0, t) = \psi(b, t) = 0, \quad (32)$$

and initial condition

$$\psi(r, t_0) = Z_0(r), \tag{33}$$

are given by the coupled set of Fredholm's integral equations

$$u_i(\tau)=\Delta_i(\tau)-k_0^{-1} \int_{t_0}^{t_1} \sum_{j=1}^{M} \hat{K}_{ij}(\tau; \alpha)\, u_j(\alpha)d\alpha, \tag{34}$$

$i=1, 2, \ldots, M$

with the kernels defined by

$$\hat{K}_{ij}(\tau, \alpha)= \frac{\phi_M^2 4V^2}{b^2} \sum_{n=1}^{\infty} \sin(\frac{n\pi r_i}{b})\sin(\frac{\pi r_i}{b})\sin(\frac{\pi r_j}{b})\sin(\frac{n\pi r_j}{b})K_n(\tau; \alpha), \tag{35}$$

in which the function $K_n(\tau; \alpha)$ is defined by relations (58) in Chapter III. The forcing functions $\Delta_i(\tau)$ are given by

$$\Delta_i(\tau) = \sum_{n=1}^{\infty} f_n(\tau)\sin(\frac{n\pi r_i}{b})\sin(\frac{\pi r_i}{b}) \int_o^b \sin(\frac{n\pi r'}{b})Z_0(r')dr', \tag{36}$$

where

$$f_n(\tau) = k_0^{-1}\, \phi_M 2V[e^{\lambda_n(\tau-t_0)} - e^{\lambda_n(2t_1-t_0-\tau)}]\, \lambda_n^{-1}, \tag{37}$$

and

$$\lambda_n = [v\Sigma_f-\Sigma_a - \frac{n^2\pi^2}{b^2} D]V. \tag{38}$$

If the control planes $r=r_i$ $i=1, 2, \ldots, M$ are placed symmetrically with respect to the center plane of the reactor; $r=b/2$, one finds that the core would have a symmetry group g composed of two symmetry transformations: the identity, I, and the mirror reflection with respect to the center plane, σ.

That $\{I, \sigma\}$ satisfies the properties of a group is clear from the relations

$$II=I \tag{39}$$

$$I\sigma=\sigma I=\sigma \tag{40}$$

and

$$\sigma\sigma=I \tag{41}$$

This group is identified in mathematical physics as the symmetry group c_s. (Reference [14]; Chapter III, describes the group classification system used by chemists. Although the classification of symmetry groups is irrelevant to the present discussion, its use may simplify the process of obtaining from compiled tables the properties of a given group).

The group g has the two unidimensional irreducible representations given below

	I	σ
Γ_1	1	1
Γ_2	1	-1

The projection operators $P_n^{(q)}$ are in this case given by

$$P_1^{(1)} = \frac{1}{2}[I+\sigma] \tag{42}$$

and

$$P_1^{(2)} = \frac{1}{2}[I-\sigma] . \tag{43}$$

For simplicity, suppose there are only two control rods (planes) symmetrically located with respect to the center plane. Then it follows that the kernels $\hat{K}_{ij}$ in the optimality conditions (34) would satisfy in this case the relations

$$\hat{K}_{12}(\tau; \alpha) = \hat{K}_{21}(\tau; \alpha) \tag{44}$$

and

$$\hat{K}_{11}(\tau; \alpha) = \hat{K}_{22}(\tau; \alpha) . \tag{45}$$

Also, since the effect of applying the transformation σ on the reactor is felt upon the optimality conditions as the interchange of subscripts 1 and 2 it follows that the kernels $\hat{K}_{ij}$ are invariant to σ. Therefore, the reduction of the optimality conditions into two decoupled equations by means of symmetry considerations is possible.

With only two control functions, $u_1(t)$ and $u_2(t)$, the space E becomes the two-dimensional Euclidean space. Applying the projections $P_n^{(q)}$ to any base in E one finds that the invariant manifolds $E_1^{(1)}$ and $E_1^{(2)}$ are in this case the unidimensional subspaces spanned by the normalized vectors $\rho_1^{(1)} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$ and $\rho_1^{(2)} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$ respectively. Thus, by expanding the control vector U in the form

$$U(t) = \sum_{q=1}^{2} \beta_1^{(q)}(t)\, \rho_1^{(q)} , \tag{46}$$

where the $\{\beta_1^{(q)}\}$ are the time-dependent expansion coefficients, and substituting (46) in the optimality conditions one finds after taking the inner products in E that the conditions for optimality become

$$\beta_1^{(1)}(\tau) = \frac{1}{\sqrt{2}} [\Delta_1(\tau)+\Delta_2(\tau)]-k_0^{-1} \int_{t_0}^{t_1} [\hat{K}_{11}(\tau;\, \alpha)$$

$$+ \hat{K}_{21}(\tau;\, \alpha)]\beta_1^{(1)}(\alpha)d\alpha \tag{47}$$

and

$$\beta_1^{(2)}(\tau) = \frac{1}{\sqrt{2}} [\Delta_1(\tau)-\Delta_2(\tau)]-k_0^{-1} \int_{t_0}^{t_1} [\hat{K}_{11}(\tau;\, \alpha)$$

$$- \hat{K}_{21}(\tau;\, \alpha)]\beta_1^{(2)}(\alpha)d\alpha \tag{48}$$

These equations can be solved independently. Given the simplicity of the present example the decoupling of the optimality conditions could have been achieved perhaps by other than the sophisticated group theoretic techniques.

However, the reduction of mathematical models in the case of reactors with 2 or 3-simensional configurations becomes more difficult and it is in this case that the benefits of applying a systematic approach are welcome. The next example illustrates the application of the symmetry principles to a three-dimensional reactor model.

5. An Example: Reduction of a Cylindrical Reactor Model

Consider the homogeneous cylindrical reactor of Figure 7 and the one-energy-group model

$$\frac{\partial \psi}{\partial t}(r, \theta, z, t) = VD \nabla^2 \psi(r, \theta, z, t) + V[\nu\Sigma_f - \Sigma_a]\psi(r, \theta, z, t)$$

$$- B(r, \theta, z)\, U(t), \tag{49}$$

where

$$B(r, \theta, z)\, U(t) = V \phi_0(r, \theta, z) \sum_{i=1}^{14} \delta(z - z_i)\, \alpha_i(r, \theta)\, u_i(t), \tag{50}$$

and

$$\phi_0(r, \theta, z) = \phi_M \cos\left(\frac{\pi}{2} z\right) J_0(\gamma_{01}\, r). \tag{51}$$

The function $\alpha_i(r, \theta)$ is equal to 1 in the small region occupied by the ith neutron-control device and equal to zero elsewhere. ∇^2 denotes the Laplacian operator in cylindrical coordinates, L denotes the length of the core. γ_{01} represents the first root of the zero-order Bessel's function $J_0(\gamma R)$, and R is the radius of the cylinder. All the other variables have the usual meaning.

The flux is equal to zero at the boundary of the core and the initial condition is given by

$$\psi(r, \theta, z, t_0) = Z_0(r, \theta, z). \tag{52}$$

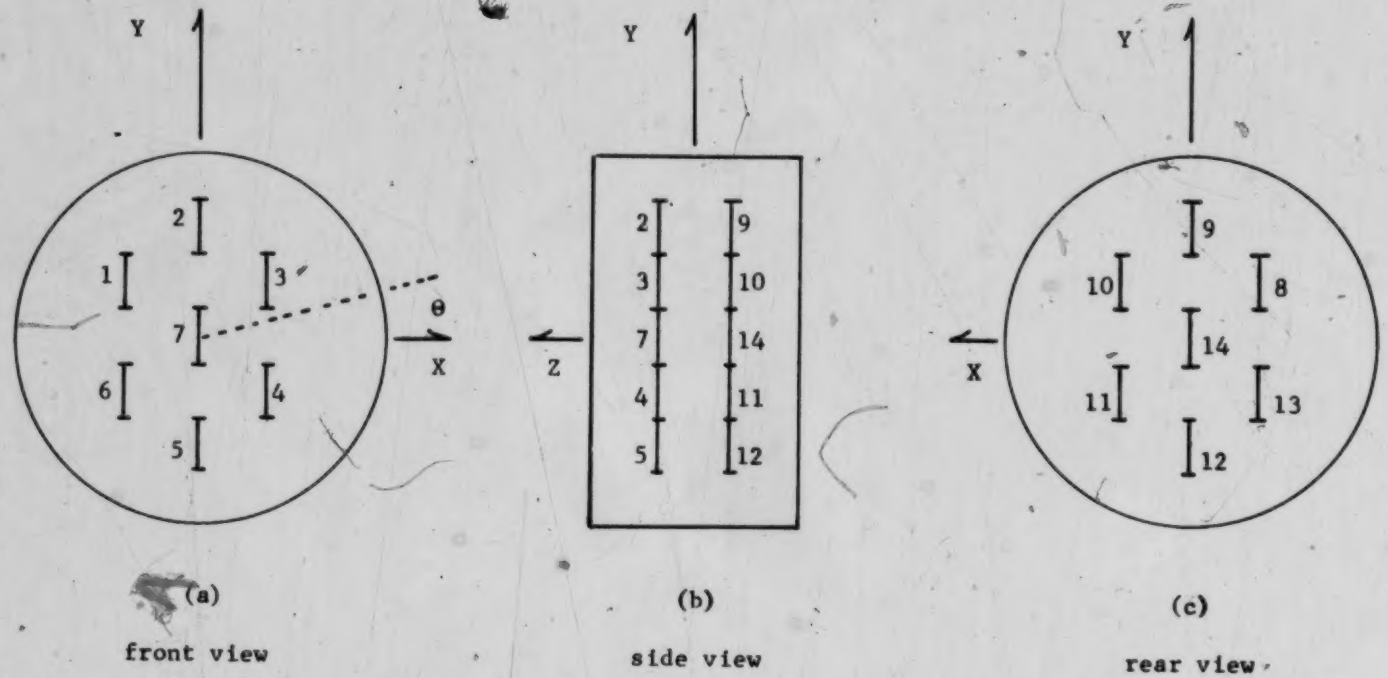


Figure 7 Cylindrical Reactor Core and Layout of Controllers.

Given the symmetrical allocation of the control devices in the core, as shown in Figure 7, one may conclude that the symmetry principles discussed in the previous sections can be applied to the problem of reducing the mathematical model.

The reactor core has the following group of symmetry operations: the identity I, anticlockwise rotation of π radians with respect to the x, y and z-axis; denoted respectively by R_x, R_y and R_z.

Reflection with respect to xy, yz and zx-planes; denoted respectively by F_{xy}, F_{yz} and F_{zx}. And finally, the inversion operation with respect to the center, R_I.

These symmetry transformations, together with the group multiplication table shown in Table 4, constitute the symmetry group D_{2h} of mathematical physics. Each of the symmetry operations forms a class in itself. The eight unidimensional irreducible representations of D_{2h} are given for reference purposes in Table 5. It follows from this observation that the mathematical model of the core can be reduced into eight subsystems by the projection operators $P_1^{(q)}$; q=1, 2, ..., 8.

In this particular example, the space E of control vectors becomes the 14-dimensional Euclidean space. The vector $U = col[u_1, \ldots, u_{14}]$ transforms under the symmetry operations as shown in Table 6. For simplicity only the subscript indices are shown. The Table shows for example that the 6-th and 11-th entries in U are interchanged by the operation R_y.

By applying the projection operators $P_1^{(q)}$ to any complete set of vectors in E, one finds that the eight invariant subspaces $\{E_1^{(q)}\}$, q=1, 2, ..., 8 are spanned by the vectors $\{\rho_{1_i}^{(q)}\}$ shown in Table 7.

D_{2h}	I	R_I	R_x	R_y	R_z	F_{zx}	F_{xy}	F_{yz}
I	I	R_I	R_x	R_y	R_z	F_{zx}	F_{xy}	F_{yz}
R_I	R_I	I	F_{yz}	F_{zx}	F_{xy}	R_y	R_z	R_x
R_x	R_x	F_{yz}	I	R_z	R_y	F_{xy}	F_{zx}	R_I
R_y	R_y	F_{zx}	R_z	I	R_x	R_I	F_{yz}	F_{xy}
R_z	R_z	F_{xy}	R_y	R_x	I	F_{yz}	R_I	F_{zx}
F_{zx}	F_{zx}	R_y	F_{xy}	R_I	F_{yz}	I	R_x	R_z
F_{xy}	F_{xy}	R_z	F_{zx}	F_{yz}	R_I	R_x	I	R_y
F_{yz}	F_{yz}	R_x	I	F_{xy}	F_{zx}	R_z	R_y	I

Table 4. Group Multiplication Table. Sample: $R_I R_x = F_{yz}$

D_{2h}	I	R_z	R_y	R_x	R_I	F_{xy}	F_{xz}	F_{yz}
$\Gamma^{(1)}$	1	1	1	1	1	1	1	1
$\Gamma^{(2)}$	1	1	1	1	-1	-1	-1	-1
$\Gamma^{(3)}$	1	1	-1	-1	1	1	-1	-1
$\Gamma^{(4)}$	1	1	-1	-1	-1	-1	1	1
$\Gamma^{(5)}$	1	-1	1	-1	1	-1	1	-1
$\Gamma^{(6)}$	1	-1	1	-1	-1	1	-1	1
$\Gamma^{(7)}$	1	-1	-1	1	1	-1	-1	1
$\Gamma^{(8)}$	1	-1	-1	1	-1	1	1	-1

Table 5. Irreducible Representations of D_{2h}

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}	u_{13}	u_{14}
I	1	2	3	4	5	6	7	8	9	10	11	12	13	14
R_z	4	5	6	1	2	3	7	11	12	13	8	9	10	14
R_y	10	9	8	13	12	11	14	3	2	1	6	5	4	7
R_x	13	12	11	10	9	8	14	6	5	4	3	2	1	7
R_I	11	12	13	8	9	10	14	4	5	6	1	2	3	7
F_{xy}	8	9	10	11	12	13	14	1	2	3	4	5	6	7
F_{zx}	6	5	4	3	2	1	7	13	12	11	10	9	8	14
F_{yz}	3	2	1	6	5	4	7	10	9	8	13	12	11	14

Table 6. Transformation of the Vector U.

Sample: $R_y u_6 = u_{11}$

$E_1^{(1)}$			$E_1^{(2)}$	$E_1^{(3)}$	$E_1^{(4)}$			$E_1^{(5)}$	$E_1^{(6)}$		$E_1^{(7)}$		$E_1^{(8)}$
$\rho_{11}^{(1)}$	$\rho_{12}^{(1)}$	$\rho_{13}^{(1)}$	$\rho_{11}^{(2)}$	$\rho_{11}^{(3)}$	$\rho_{11}^{(4)}$	$\rho_{12}^{(4)}$	$\rho_{13}^{(4)}$	$\rho_{11}^{(5)}$	$\rho_{11}^{(6)}$	$\rho_{12}^{(6)}$	$\rho_{11}^{(7)}$	$\rho_{12}^{(7)}$	$\rho_{11}^{(8)}$
1	0	0	1	1	1	0	0	1	1	0	1	0	1
0	1	0	0	0	0	1	0	0	0	0	0	1	0
1	0	0	-1	-1	1	0	0	-1	1	1	1	0	-1
1	0	0	1	1	1	0	0	-1	-1	-1	-1	0	-1
0	1	0	0	0	0	1	0	0	0	0	0	-1	0
1	0	0	-1	-1	1	0	0	1	-1	-1	-1	0	1
0	0	1	0	0	0	0	1	0	0	0	0	0	0
1	0	0	-1	1	-1	0	0	-1	1	-1	-1	0	1
0	1	0	0	0	0	-1	0	0	0	0	0	-1	0
1	0	0	1	-1	-1	0	0	1	1	-1	-1	0	-1
1	0	0	-1	1	-1	0	0	1	-1	1	1	0	-1
0	1	0	0	0	0	-1	0	0	0	0	0	1	0
1	0	0	1	-1	-1	0	0	-1	-1	1	1	0	1
0	0	1	0	0	0	0	-1	0	0	0	0	0	0

Table 7. The Invariant Subspaces $E_1^{(q)}$ and the Base-Vectors $\rho_{1m}^{(q)}$

Also, the eigenfunctions of the spatial operator

$$VD\nabla^2 + V[\nu\Sigma_f - \Sigma_a] \tag{56}$$

generate all the irreducible representations of the group D_{2h}. Table 8 shows the classification of the eigenfunctions according to the irreducible representation which they generate. The set of eigenfunctions is orthogonal in the real Hilbert space H, endowed with the inner product

$$\langle \psi, \phi \rangle = \int_{-L/2}^{L/2} \int_{0}^{2\pi} \int_{0}^{2} r\, \psi(r,\theta,z)\, \phi(r,\theta,z)\, dr d\theta dz. \tag{57}$$

By applying the method of eigenfunction expansions one can find that the component $\psi_1^{(q)}(r, \theta, z, t)$ of the flux $\psi(r, \theta, z, t)$, which lies in the invariant subspace $H_1^{(q)}$ spanned by the infinite set $\{\psi_{nik}^{(q)}\}$, is given by

$$\psi_1^{(q)}(r, \theta, z, t) = \sum_{(nik)\in\Omega^{(q)}} \psi_{nik}^{(q)}(r, \theta, z) \left[\langle Z_0, \psi_{nik}^{(q)} \rangle_H e^{\lambda_{nik}(t-t_0)} \right.$$

$$\left. + \sum_{m=1}^{N_q} \int_{t_0}^{t} \langle B\, \rho_{lm}^{(q)}, \psi_{nik}^{(q)} \rangle_H e^{\lambda_{nik}(t-\tau)}\, \beta_{lm}^{(q)}(\tau) d\tau \right] \tag{58}$$

where $\Omega^{(q)}$ denotes the infinite set of subscripts associated with the eigenfunctions in $H_1^{(q)}$, and $\lambda_{nik}^{(q)}$ denotes the eigenvalue associated with $\psi_{nik}^{(q)}$,

$$\lambda_{nik}^{(q)} = - VD[\gamma_{lk}^2 + \alpha_{nik}^{(q)2}] + [\nu\Sigma_f - \Sigma_a]\, V, \tag{59}$$

where

Subsystem q	Laplacian Modes $\psi^{(q)}_{nik}(r, \theta, z)$	
1	$f_{ik}\text{Cos}((2n-1)\frac{\pi}{L}z)\ \cos(i\theta)\ J_i(\gamma_{ik}r)$	i=0, 2, 4, ..
2	$f_{ik}\text{Sin}(2n\frac{\pi}{L}z)\ \ \text{Sin}(i\theta)\ J_i(\gamma_{ik}r)$	i=2, 4, 6, ..
3	$f_{ik}\text{Cos}((2n-1)\frac{\pi}{L}z)\ \text{Sin}(i\theta)\ J_i(\gamma_{ik}r)$	i=2, 4, 6, ..
4	$f_{ik}\text{Sin}(2n\frac{\pi}{L}z)\ \ \text{Cos}(i\theta)\ J_i(\gamma_{ik}r)$	i=0, 2, 4, ..
5	$f_{ik}\text{Sin}(2n\frac{\pi}{L}z)\ \ \text{Cos}(i\theta)\ J_i(\gamma_{ik}r)$	i=1, 3, 5, ..
6	$f_{ik}\text{Cos}((2n-1)\frac{\pi}{L}z)\ \text{Sin}(i\theta)\ J_i(\gamma_{ik}r)$	i=1, 3, 5, ..
7	$f_{ik}\text{Sin}(2n\frac{\pi}{L}z)\ \ \text{Sin}(i\theta)\ J_i(\gamma_{ik}r)$	i=1, 3, 5, ..
8	$f_{ik}\text{Cos}((2n-1)\frac{\pi}{L}z)\ \text{Cos}(i\theta)\ J_i(\gamma_{ik}r)$	i=1, 3, 5, ..

Where

$n = 1, 2, 3, ..$ $\quad k = 1, 2, 3, ..$

$-\frac{L}{2} \le z \le \frac{L}{2}$ $\quad 0 \le \theta \le 2\pi$

γ_{ik} is the kth root of $J_i(\gamma R)$

$$f_{ik} = \begin{cases} 2/(\sqrt{\pi L}\ R\ J_{i+1}(\gamma_{ik}R)) & \text{if } i \neq 0 \\ \sqrt{2}\ /\ (\sqrt{\pi L}\ R\ J_{i+1}(\gamma_{ik}R)) & \text{if } i = 0 \end{cases}$$

Table 8 The Laplacian Eigenfunctions $\psi^{(q)}_{nik}$ —

$$\alpha_{nik}^{(q)} = (2n-1)\frac{\pi}{L} \quad , \quad \text{for } q=1,\ 3,\ 6,\ 8 \tag{60}$$

or

$$\alpha_{nik}^{(q)} = \frac{2n\pi}{L} \quad , \text{ for } q=2,\ 4,\ 5,\ 7. \tag{61}$$

$\beta_1^{(q)}$ denotes the expansion coefficient corresponding to the base-vector $\rho_{lm}^{(q)}$ in the expansion of the control U,

$$U(\tau) = \sum_{q=1}^{8} \sum_{m=1}^{N_q} \beta_{lm}^{(q)}(\tau)\, \rho_{lm}^{(q)} \quad , \tag{62}$$

and N_q is the dimension of the invariant subspace $E_1^{(q)}$.

While the dimension of the control space E in the original model is 14, the dimension of the largest subspaces, $E_1^{(1)}$ and $E_1^{(4)}$, is only 3.

CHAPTER VI

A MODAL EXPANSION APPROACH TO LOAD-FOLLOWING

1. Introduction

Given the large difference in time constants between neutron kinetics and reactor poisoning dynamics, the analysis of problems such as xenon spatial stability or load-following can be simplified considerably by making use of the valid assumption that the neutron kinetics is an instantaneous process, compared to the slow dynamics of the core poisoning effects.

With this assumption a very general reactor core model would consist of the multigroup neutron diffusion equations at steady state, coupled with the xenon and iodine dynamic equations.

A new method is presented in this chapter, for obtaining an approximate solution to such a model. The objective is to obtain an explicit functional relation between the neutron flux and the neutron absorption cross sections of the zone controllers. The motive is to reduce the number of variables involved in the optimal control studies of load following operations.

The proposed approach is described in terms of the one-energy group model with xenon and iodine dynamics. Several examples are presented in which use is made of data that corresponds to a typical, pressurised tube, heavy water-moderated, large nuclear reactor.

2. Background

The variety of reactor models used in connection with the control of distributed nuclear reactors are usually classified as either nodal [1,2,3], finite-difference [4,5,6] or modal expansion [7,8,9] depending on which method is used to obtain an approximate description of the reactor core processes.

Often a method classified as one of the above is further categorized depending on the details and specific features of the approach. A modal approach could be categorized for example by the type of modal expansion used. A description and comparison of the basic approaches to distributive reactor modeling is given in reference [10].

The nodal and modal expansion methods are, by far, the most widely employed approaches to reactor control problems in which use is made of distributed reactor models [11-20]. The method of finite-differences is used extensively and almost exclusively in reactor physics calculations.

The main advantages of the modal expansion approach over the finite-difference and nodal methods can be listed as follows:

a) The modal expansion method is capable of representing the distributed nature of the reactor core in the same detail as would be obtained with the finite-difference approximation. The number of equations required is, however, much less for the modal expansion approach.

b) The number of equations required to represent the distributed reactor is comparable for the modal expansion and nodal methods. The spatial representation of the reactor core is however more detailed for the modal expansion approach.

In general, the main disadvantages of the modal approach are the following:

a) For many modal expansion methods it is not possible to determine or even estimate error bounds between the true solution to the problem and the solution obtained by means of the modal expansion.

b) For some modal expansion methods the spatial modes are difficult to compute [10,15].

Several methods which make use of explicitly known, analytical functions as expansion modes have been proposed in the past to circumvent these disadvantages. The method of solution functions of R. Bobone [21-23] is probably the best example. The method was used successfully in criticality calculations and also in computing the steady state power distribution in a reactor core. His approach assumes the homogeneity of the diffusion coefficient and reactor buckling within each of several core regions. The method consists of expanding the neutron flux within each region in terms of solutions to the diffusion equation and evaluating the expansion coefficients so as to minimize the mean squared error of the neutron flux and current at the region interfaces. This method, unfortunately, cannot be applied to the analysis of load following operations, since the xenon distribution within the reactor regions makes the neutron absorption cross sections not only spatially but also temporally dependent. Iwazumi and Koga [18] proposed a method in which the spatially dependent parameters of the diffusion equation are replaced with constant terms that are chosen to keep the fundamental mode steady. The spatially dependent terms were then treated as the initial distribution of the control variable. The basic idea was to use the eigenfunctions of the Helmholtz operator as expansion modes. This approach was used by Iwazumi and Koga in connection with the problem of changing the flux

distribution from a given initial state to a desired final state in a short period of time while minimizing a functional that penalizes the terminal flux distribution error and the deviations of the control variable from the initial control distribution. The one-energy group model with one delayed neutron precursor in a slab reactor was considered in their work. Recently, Iwazumi's method was used by Asatani et. al. [19] in connection with the same control problem and the same reactor model.

Iwazumi's method has not been extended to treat the load following or xenon oscillation problems. Whether or not this extension would lead to justifiable simplifications is an open question.

3. The Approach

The method presented in this chapter can be classified as a modal expansion approximation in terms of analytical functions and it is characterized by two important features:

1) The use of analytical functions as expansion modes makes it possible to bypass the problem of computing the expansion modes by numerical means.

2) The formalism of functional analysis, which is used in the development of the approximate model, makes it possible to estimate bounds on the approximating errors.

Three basic assumptions are considered:

1) The geometrical configuration of the reactor core is such that a complete set of wavefunctions in $L_2[V]$ is explicitly known and has the property that the functions vanish at the extrapolated boundary of the core. $L_2[V]$ is the space of squared integrable functions over the core volume V.

2) The neutron diffusion coefficients for the different neutron energies are assumed to be homogeneous throughout the core. This assumption is acceptable for large reactor cores that in addition to being lightly rodded either do not make use of reflectors [36] or have reflectors made of the same material employed as neutron moderator [36]. The multigroup bucklings may be spatially dependent.

3) The reactor model is linearized in the neighborhood of a given power distribution which could either be at steady state or also be a slow transient during a large power level change.

The description of the proposed approach is given in terms of the

one-group model with xenon and iodine dynamics presented in Table 9. For the sake of clarity, the procedure has been divided into five steps:

1) The neutron diffusion equation (1) is transformed into a nonhomogeneous Fredholm's integral operator equation of the second kind [24-29].

2) The kernel of the resulting integral operator is approximated by a degenerate kernel. An error bound on the $L_2[V]$ norm is obtained, of the error between the true solution to the Fredholm's equation and the solution obtained by means of the degenerate kernel approximation.

3) The integral equation is solved for the neutron flux by applying the method of degenerate kernels [24-29]. An explicit relation for the neutron flux is obtained in terms of the xenon concentration and the control variables.

4) The explicit expression for the neutron flux is substituted in equations (5) and (6) and an approximate solution to these equations is obtained by expanding the xenon and iodine distributions in terms of a finite number of wavefunctions. A bound is found on the $L_2^2[V]$ norm of the error between the true solution to equations (5) and (6) and the solution obtained by means of the modal approximation. The bound is obtained by showing that the error satisfies a nonhomogeneous Volterra's equation of the second kind in the product space $L_2^2[V]$ and then using the Neumann series representation for the error.

5) Using the explicit expression for the neutron flux and the modal expansion solution to equations (5) and (6), the neutron flux is expressed in terms of the control variables only.

Table 9. One-Energy Neutronic Model with Xenon and Iodine Dynamics

The model describes the dynamic behavior of the neutron flux deviation ψ_ϕ, and the deviation of xenon and iodine concentrations from their corresponding equilibrium distributions ϕ_0, X_0 and I_0. The parameters σ, λ_x, λ_I and γ_I denote, respectively: the neutron absorption microscopic cross section of xenon-135, the decay constants of xenon and iodine, and the iodine yield per fission. $\hat{u}$ denotes the control-function deviations from the equilibrium control u_0.

$$\nabla^2\psi_\phi(\underline{r}, t) + b(\underline{r})\psi_\phi(\underline{r}, t) = f(\underline{r}, t) \qquad r\varepsilon V \qquad (1)$$

$$b(\underline{r}) = [v\Sigma_f(\underline{r}) - \Sigma_a(\underline{r}) - u_o(\underline{r}) - \sigma x_o(\underline{r})]/D \qquad (2)$$

$$f(\underline{r}, t) = \frac{\sigma}{D}\phi_o(\underline{r})\psi_x(\underline{r}, t) + \hat{u}(\phi_o(\underline{r}), \underline{r}, t) \qquad (3)$$

$$x_o(\underline{r}) = \frac{\gamma_I \Sigma_f(\underline{r})\ \phi_o(\underline{r})}{\lambda_x + \sigma\ \phi_0(\underline{r})} \qquad (4)$$

$$\frac{\partial\psi_I}{\partial t}(\underline{r}, t) = -\lambda_I\psi_I(\underline{r}, t) + \gamma_I\ \Sigma_f(\underline{r})\ \psi_\phi(\underline{r}, t) \qquad (5)$$

$$\frac{\partial\psi_x}{\partial t}(\underline{r}, t) = \lambda_I\psi_I(\underline{r}, t) - [\lambda_x + \sigma\phi_o(\underline{r})]\psi_x(\underline{r}, t) - x_o(\underline{r})\,\sigma\ \psi_\phi(\underline{r}, t) \qquad (6)$$

Initial Conditions:

$$\psi_I(\underline{r}, t_0) = \psi_{I0}(\underline{r}) \qquad (7)$$

$$\psi_x(\underline{r}, t_0) = \psi_{x0}(\underline{r}) \qquad (8)$$

Boundary condition at the core extrapolated boundary (∂V):

$$\psi_\phi(\underline{r}, t) = 0 \qquad (9)$$

4. Solving the Diffusion Equation

The one-group neutron diffusion equation (1) can be seen to be a nonhomogeneous Helmholtz equation in the Hilbert space $L_2[V]$ with a space-dependent parameter $b(\underline{r})$, which by assumption is a piecewise continuous function of the spatial variable $\underline{r}$. An approximate solution to equation (1) could be obtained by expanding the neutron flux in terms of a finite number of functions. In fact this procedure is used extensively in the literature. The simplicity of this approach however, has to be weighted against the disadvantage of lacking error estimates. An alternate approach consists of transforming the equation (1) into an equivalent integral equation that is amenable to both error analysis and computation.

The first step toward transforming (1) is to express $b(\underline{r})$ as the sum of two terms,

$$b(\underline{r}) = b_0 + \hat{b}(\underline{r}) \tag{10}$$

where b_0 is spatially independent. There are of course an infinite number of different ways of doing this. The considerations that dictate the best choice will be discussed later. For the moment it is assumed that b_0, in addition to being a real number, is also different from all the eigenvalues of the following boundary value problem in $L_2[V]$:

$$\nabla^2 \psi_j(\underline{r}) = \lambda_j \psi_j(\underline{r}) , \qquad \underline{r} \in V \tag{11}$$

and

$$\psi_j(\underline{r}) = 0 \quad , \quad \underline{r}\varepsilon\partial V \tag{12}$$

where V is the region occupied by the reactor core, ∂V is the extrapolated boundary and ∇^2 is the Laplacian operator. By assumption the set of functions $\{\psi_j\}$ is explicitly known.

Using (10) in equation (1) one finds that

$$\nabla^2\psi_\phi(\underline{r}, t) + b_0\,\psi_\phi(\underline{r}, t) = f(\underline{r}, t) - \hat{b}(\underline{r})\,\psi_\phi(\underline{r}, t) \tag{13}$$

and using the Green's function $G(\underline{r}, \underline{r}')$ corresponding to the left side of equation (13) the following expression is obtained:

$$\psi_\phi(\underline{r}, t) = \int_V G(\underline{r}, \underline{r}')\, f(\underline{r}')dV' - \int_V G(\underline{r}, \underline{r}')\,\hat{b}(\underline{r}')\,\psi_\phi(\underline{r}', t)dV' \tag{14}$$

where the integrals are over the core volume.

By invoking the first assumption of section 3 it is not difficult to show that the Green's function can be obtained in series form;

$$G(\underline{r}, \underline{r}') = \sum_j \frac{\psi_j(\underline{r})\,\psi_j(\underline{r}')}{(b_0 + \lambda_j)} \tag{15}$$

where $\{\lambda_j\}$ and $\{\psi_j\}$ form the eigensolution to equation (11).

Equation (14) is a Fredholm's integral equation of the second kind in the function space $L_2[V]$. Also, since the integral operator in (14) is compact, it is clear that there exists a unique solution in $L_2[V]$ provided that the number (-1) is not in the eigenvalue spectrum of the integral operator [38].

4.1 The Method of Degenerate Kernels

Of all the methods available for solving the nonhomogeneous Fredholm's integral equation, the method of degenerate kernels [24-29] is particularly attractive to use when the Green's function (15) can be adequately approximated by a finite series.

Substituting the Green's function (15) in the integral equation (14) the resulting expression is

$$\psi_\phi(\underline{r}, t) = \sum_j \frac{\psi_j(\underline{r})}{(b_0 + \lambda_j)} \left[\int_V \psi_j(\underline{r}')(f(\underline{r}', t) - \hat{b}(\underline{r}')\psi_\phi(\underline{r}', t))dV' \right] \tag{16}$$

and can be rewritten in the form;

$$\psi_\phi(\underline{r}, t) = \sum_j \frac{\psi_j(\underline{r})}{(b_0 + \lambda_j)} [e_j(t) - C_j(t)] \tag{17}$$

where the coefficients $e_j(t)$ and $C_j(t)$ are defined as follows;

$$C_j(t) = \int_V \psi_j(\underline{r})\, \hat{b}(\underline{r})\, \psi_\phi(\underline{r}, t)\, dV \tag{18}$$

$$e_j(t) = \int_V \psi_j(\underline{r})\, f(\underline{r}, t) dV \tag{19}$$

and satisfy the algebraic system of equations;

$$C_i(t) = \sum_j d_{ij}\, (e_j - C_j) \tag{20}$$

where

$$d_{ij} = \int_V \frac{\psi_i(\underline{r})\, \hat{b}(\underline{r})\, \psi_j(\underline{r}) dV}{(b_0 + \lambda_j)} \tag{21}$$

A natural approach to solving equation (16) consists of approximating the Green's function by a finite series and then solving the algebraic equation (20). It is clear that the approximation can be made as accurate as desired by just including more terms in the series. In addition to the computational simplicity of this approach the method also has the property of being amenable to error analysis.

The objective in this section is to determine an error bound between the true and the approximate solutions to the equation (16).

It is convenient to introduce the following notation; $\tilde{G}(\underline{r}, \underline{r}')$ denotes the approximate Green's function and Ω denotes the finite set of subscripts $\{j\}$ corresponding to the terms that appear in the series representation of $\tilde{G}(\underline{r}, \underline{r}')$. That is;

$$\tilde{G}(\underline{r}, \underline{r}') \triangleq \sum_{j\in\Omega} \frac{\psi_j(\underline{r})\,\psi_j(\underline{r}')}{(b_0+\lambda_j)} \tag{22}$$

Also, if ϕ is a function in $L_2[V]$, then L, $\tilde{L}$, $\hat{L}$ and $\tilde{\hat{L}}$ denote the following transformations in $L_2[V]$;

$$L\phi \triangleq \int_V G(\underline{r}, \underline{r}')\phi(\underline{r}')dV' \tag{23}$$

$$\tilde{L}\phi \triangleq \int_V \tilde{G}(\underline{r}, \underline{r}')\phi(\underline{r}')dV' \tag{24}$$

$$\hat{L}\phi \triangleq Lb\phi = \int_V G(\underline{r}, \underline{r}')\hat{b}(\underline{r}')\phi(\underline{r}')dV' \tag{25}$$

and

$$\tilde{\hat{L}}\phi \triangleq \tilde{L}\hat{b}\phi = \int_V \tilde{G}(\underline{r}, \underline{r}')\hat{b}(\underline{r}')\phi(\underline{r}')dV' \tag{26}$$

With these definitions, the equation (16) and its approximate

version are written in operator form;

$$\psi_\phi(t) = L\, f(t) - \hat{L}\, \psi_\phi(t) \tag{27}$$

and

$$\tilde{\psi}_\phi(t) = \tilde{L}\, f(t) - \tilde{\hat{L}}\, \tilde{\psi}_\phi(t) \tag{28}$$

where ψ_ϕ, $\tilde{\psi}_\phi$ and f are shown as functions of t to emphasize their temporal dependency.

From equations (27) and (28) the error $\varepsilon(t) = \psi_\phi(t) - \tilde{\psi}_\phi(t)$ is given by the expression;

$$\varepsilon(t) = (L-\tilde{L})\, f(t) - [\hat{L}\, \psi_\phi(t) - \tilde{\hat{L}}\, \tilde{\psi}_\phi(t)] \tag{29}$$

which can be rewritten as follows

$$\varepsilon(t) = (L-\tilde{L})\, f(t) - \hat{L}\, \varepsilon(t) - (\hat{L}-\tilde{\hat{L}})\tilde{\psi}_\phi(t) \tag{30}$$

Hence the $L_2[V]$ norm of $\varepsilon(t)$ is bounded above by

$$||\varepsilon(t)|| \leq ||(L-\tilde{L})f(t)|| + ||\hat{L}||\ ||\varepsilon(t)|| + ||(\hat{L}-\tilde{\hat{L}})\tilde{\phi}_\phi(t)|| \tag{31}$$

and it follows that if $\hat{L}$ is a contraction operator, then the error estimate is obtained in the form

$$||\varepsilon(t)|| \leq \frac{||(L-\tilde{L})f(t)||}{[1-||\hat{L}||]} + \frac{||(\hat{L}-\tilde{\hat{L}})\tilde{\psi}_\phi(t)||}{[1-||\hat{L}||]} \tag{32}$$

Given that the set $\{\psi_j\}$ is complete in the Hilbert space $L_2[V]$, every function ϕ in $L_2[V]$ has a fourier series representation

$$\phi = \sum_j \psi_j \langle \psi_j, \phi \rangle \tag{33}$$

where $\langle \psi_j, \phi \rangle$ denotes the inner product in $L_2[V]$;

$$\langle \psi_j, \phi \rangle = \int_V \psi_j(\underline{r})\, \bar{\phi}(\underline{r}) dV \tag{34}$$

and $\bar{\phi}$ is the complex conjugate of ϕ. Also, using the inner product notation it can be seen that

$$(L-\tilde{L})\, f(t) = \sum_{j \notin \Omega} \frac{\psi_j \langle \psi_j, f(t) \rangle}{(b_0 + \lambda_j)} \tag{35}$$

and given the orthonormality of the functions ψ_j:

$$(L-\tilde{L}) f(t) = \sum_{j \notin \Omega} \frac{\psi_j}{(b_0 + \lambda_j)} \langle \psi_j, (f(t) - \sum_{i \in \Omega} \psi_i \langle \psi_i, f(t) \rangle) \rangle \tag{36}$$

$$= (L-\tilde{L})(f(t) - \sum_{i \in \Omega} \psi_i \langle \psi_i, f(t) \rangle) \tag{37}$$

Similarly using the relations $\hat{L} = Lb$ and $\hat{\tilde{L}} = \tilde{L}b$, it follows that

$$(\hat{L}-\hat{\tilde{L}})\tilde{\psi}_\phi(t) = (L-\tilde{L})(b\tilde{\psi}_\phi(t) - \sum_{i \in \Omega} \psi_i \langle \psi_i, b\tilde{\psi}_\phi(t) \rangle) \tag{38}$$

Substituting (37) and (38) in the inequality (32), the error estimate is finally obtained:

$$||\varepsilon(t)|| \leq \frac{||L-\tilde{L}||}{[1-||\hat{L}||]}\,\beta(t) \tag{39}$$

where

$$\beta(t)= ||f(t)-\sum_{i\in\Omega}\psi_i\langle\psi_i, f(t)\rangle||+||\hat{b}\tilde{\psi}_\phi(t)-\sum_{i\in\Omega}\psi_i\langle\psi_i, \hat{b}\tilde{\psi}_\phi(t)\rangle|| \tag{40}$$

The norms of the operators L, $\tilde{L}$, $\hat{L}$ and $(L-\tilde{L})$ are given by;

$$||L||= \max_j \frac{1}{|b_0+\lambda_j|} \tag{41}$$

$$||\tilde{L}||= \max_{j\in\Omega} \frac{1}{|b_0+\lambda_j|} \tag{42}$$

$$||L-\tilde{L}||= \max_{j\notin\Omega} \frac{1}{|b_0-\lambda_j|} \tag{43}$$

and

$$||\hat{L}||= \max_j \gamma_j^{1/2} \tag{44}$$

where $\{\gamma_j\}$ is the eigenvalue spectrum of $\hat{L}^*\hat{L}$ and $\hat{L}^*$ is the adjoint of $\hat{L}$. Often it is difficult to evaluate the norm of $\hat{L}$ and only an upper bound estimate can be obtained. A conservative estimate is given by;

$$||\hat{L}|| \leq \max_j \frac{1}{|b_0+\lambda_j|} \max_{\underline{r}} |\hat{b}(\underline{r})| \tag{45}$$

The derivation of these norms is presented in Appendix II.

Inequality (39) provides a useful criterion to determine how many and which terms should be used to approximate the Green's function (15) by a series expansion. It is also clear from the same inequality (39) that b_0, in the decomposition of b as shown in (10), should be chosen so as to minimize the norm of $\hat{L}$.

4.2 An Example

The data for this example has been obtained by applying the modified-one-group approximation to the two-energy group data presented in the Appendix I. The modified-one-group data and the constants corresponding to the xenon and iodine dynamics are given in Table 10 . The core configuration and the layout of zone controllers considered here is the same as in the example in Chapter V. This is shown in Figure 7. The core is cylindrical and exhibits a degree of symmetry that corresponds to the point group D_{2h} of mathematical physics [34], which has eight unidimensional, irreducible representations.

The Laplacian modes for the cylindrical case $\{\psi_{nik}\}$ are ordered with the help of three subscripts. These modes are presented in Table 8 where these have been classified according to the irreducible representation of D_{2h}. The set of eigenvalues $\{\lambda_{nik}\}$ corresponding to the Laplacian modes $\{\psi_{nik}\}$ are given in Table 11.

The parameter b_0 was chosen to be the constant value required to satisfy,

$$||b_0|| = ||b|| \qquad (46)$$

where the norm is the usual $L_2[V]$ norm. The numerical values for $||b||$,

Migration area	$M^2 = \frac{D}{\Sigma_a} = 369.7\ cm^2$
Diffusion coefficient	$D = 1.2349\ cm$
Infinite multiplication factor	$K_\infty = M^2 \frac{\nu\Sigma_f}{D} = 1.04027$
Xenon decay constant	$\lambda_x = 2.09 \times 10^{-5}\ sec^{-1}$
Iodine decay constant	$\lambda_I = 2.87 \times 10^{-5}\ sec^{-1}$
Iodine yield constant	$\gamma_I = 6.4 \times 10^{-2}$
Absorption macroscopic cross section of xenon	$\sigma = 1.22 \times 10^{-18}\ cm^2$
Neutron flux distribution at steady state	$\phi_0 = a_0 \cos(\frac{\pi}{L})z\ J_0(\gamma_{01}\ r)$
	$a_0 = 0.1110 \times 10^{15}$
Core configuration	cylindrical
Length of core	$L = 600\ cm$
Radius of core	$R = 400\ cm$
First root of $J_0(\gamma R)$	$\gamma_{01} = 6.01205 \times 10^{-5}$

Table 10 Data for the Modified One-Energy Model with Xenon and Iodine

Subsystem q	Eigenvalue $\lambda^{(q)}_{nik}$	
1	$(2n-1)^2 (\frac{\pi}{L})^2 + \gamma^2_{ik}$	i = 0, 2, 4, ..
2	$(2n)^2 (\frac{\pi}{L})^2 + \gamma^2_{ik}$	i = 2, 4, 6, ..
3	$(2n-1)^2 (\frac{\pi}{L})^2 + \gamma^2_{ik}$	i = 2, 4, 6, ..
4	$(2n)^2 (\frac{\pi}{L})^2 + \gamma^2_{ik}$	i = 0, 2, 4, ..
5	$(2n)^2 (\frac{\pi}{L})^2 + \gamma^2_{ik}$	i = 1, 3, 5, ..
6	$(2n-1)^2 (\frac{\pi}{L})^2 + \gamma^2_{ik}$	i = 1, 3, 5, ..
7	$(2n)^2 (\frac{\pi}{L})^2 + \gamma^2_{ik}$	i = 1, 3, 5, ..
8	$(2n-1)^2 (\frac{\pi}{L})^2 + \gamma^2_{ik}$	i = 1, 3, 5, ..

Table 11 The Laplacian Eigenvalues

$\|\|b\|\|$	$b_0 = \|\|b\|\|/V^{1/2}$	$\|\|b-b_0\|\|$
0.138299×10^{1}	0.796364×10^{-4}	0.261035

Table 12 The Parameter b_0 and the $L_2[V]$ norms of b and $\hat{b}$

b_0 and $||\hat{b}||$ are given in Table 12.

Two Laplacian modes per subsystem were considered to represent the Green's function (15) which in this case is of the form

$$G(r, \theta, z; r', \theta', z') = \sum_{q=1}^{8} G^{(q)}(r, \theta, z; r', \theta', z') \tag{47}$$

where

$$G^{(q)}(r, \theta, z; r', \theta', z') = \sum_{(nik)\varepsilon\Omega^{(q)}} \frac{\psi^{(q)}_{nik}(r, \theta, z)\psi^{(q)}_{(nik)}(r', \theta', z')}{[b_0 + \lambda^{(q)}_{nik}]} \tag{48}$$

where the superscript (q) corresponds to the qth subsystem. The modes in each subsystem q were selected to minimize the norm $||(L^{(q)} - \tilde{L}^{(q)})||$.

Table 13 shows the subscripts corresponding to the selected functions. The associated norms of $L^{(q)}$, $(L^{(q)} - \tilde{L}^{(q)})$ and the estimates for $||\hat{L}^{(q)}||$ are also shown in Table 13. The estimates $||\hat{L}^{(q)}||$ were evaluated from

$$||\hat{L}^{(q)}|| \leq ||L^{(q)}||b' \tag{49}$$

where b' is an effective constant such that

$$||b'|| = ||\hat{b}|| \tag{50}$$

The maximum values of $\hat{b}$ occur near the center of the core and at the extrapolated boundary. Since the flux is zero at the extrapolated boundary and also since the contribution to the norm of $||\hat{b}\phi||$ by a

Subsystem (q)	$\Omega^{(q)} = \{(nik)\}$	$L^{(q)}$ and $\tilde{L}^{(q)}$	$L^{(q)} - \tilde{L}^{(q)}$	(Estimate) $\hat{L}^{(q)}$
1	(1, 0, 1); (1, 2, 1)	0.622012×10^{5}	0.723453×10^{4}	0.934
2	(1, 2, 1); (1, 4, 1)	0.513175×10^{4}	0.214191×10^{4}	0.7714×10^{-1}
3	(1, 2, 1); (1, 4, 1)	0.887950×10^{4}	0.256095×10^{4}	0.1335
4	(1, 0, 1); (1, 2, 1)	0.151127×10^{5}	0.453571×10^{4}	0.2272
5	(1, 1, 1); (1, 3, 1)	0.821102×10^{4}	0.296172×10^{4}	0.1234
6	(1, 1, 1); (1, 3, 1)	0.252902×10^{5}	0.391550×10^{4}	0.3801
7	(1, 1, 1); (1, 3, 1)	0.821102×10^{4}	0.296172×10^{4}	0.1234
8	(1, 1, 1); (1, 3, 1)	0.252902×10^{5}	0.391550×10^{4}	0.3801

Table 13 The Norm of $L^{(q)}$, $\tilde{L}^{(q)}$ and $\hat{L}^{(q)}$

neighborhood of small radius and located in the center of the core is negligible then it can be seen that for most flux shapes of practical interest the estimate (49) is indeed valid.

The error estimates are presented in Table 14 in terms of the forcing function f and the approximate solution $\tilde{\psi}_\phi$. The method of degenerate kernels was then applied to obtain the approximate solution $\tilde{\psi}_\phi$ as shown in Table 15. The numerical values of the parameters $a_{nik}^{(q)}$, $\xi_{nik}^{(q)}$ and $d_{nik}^{(q)}$ are presented in Table 16.

$$||\varepsilon(t)|| = || \sum_{q=1}^{8} \varepsilon^{(q)}(t) ||$$

$$\leq \sum_{q=1}^{8} \frac{||(L^{(q)} - \tilde{L}^{(q)})||}{(1 - ||\hat{L}^{(q)}||)} \beta^{(q)}(t) \qquad (51)$$

where

$$\beta^{q}(t) = ||f^{(q)} - \sum_{nik \in \Omega^{(q)}} \psi_{nik}^{(q)} f_{nik}^{(q)} || + ||\hat{\tilde{b}}\tilde{\psi}_{\phi}^{(q)} - \sum_{nik \in \Omega^{(q)}} \psi_{nik}^{(q)} g_{nik}^{(q)}|| \qquad (52)$$

$$f_{nik}^{(q)} = \int_{-\frac{L}{2}}^{\frac{L}{2}} \int_{o}^{2\pi} \int_{o}^{R} \psi_{nik}^{(q)}(r, \theta, z)\, f(r, \theta, z, t)\, r\, dr\, d\theta dz \qquad (53)$$

$$g_{nik}^{(q)} = \int_{-\frac{L}{2}}^{\frac{L}{2}} \int_{o}^{2\pi} \int_{o}^{R} \psi_{nik}^{(q)}(r, \theta, z)\, b(r, \theta, z)\, \tilde{\psi}_{\phi}^{(q)}(r, \theta, z, t)\, r\, dr\, d\theta dz \qquad (54)$$

Table 14 The Error Estimate

$$\tilde{\psi}_\phi(r, \theta, z, t) = \sum_{q=1}^{8} \tilde{\psi}_\phi^{(q)}(r, \theta, z, t) \qquad (55)$$

$$\tilde{\psi}_\phi^{(q)}(r, \theta, z, t) = \sum_{(nik)\epsilon\Omega^{(q)}} a_{nik}^{(q)} \psi_{nik}^{(q)}(r, \theta, z) \int_{-\frac{L}{2}}^{\frac{L}{2}} \int_0^{2\pi} \int_0^R \psi_{nik}^{(q)}(r', \theta', z') f(r', \theta', z', t) r'dr'd\theta'dz' \qquad (56)$$

$$a_{nik}^{(q)} = \xi_{nik}^{(q)} / [1 + d_{nik}^{(q)}] \qquad (57)$$

$$\xi_{nik}^{(q)} = 1/ [b_0 + \lambda_{nik}^{(q)}] \qquad (58)$$

$$d_{nik}^{(q)} = \xi_{nik}^{(q)} \int_{-\frac{L}{2}}^{\frac{1}{2}} \int_0^{2\pi} \int_0^R \psi_{nik}^{(q)}(r', \theta', z') \hat{b}(r', \theta', z') \psi_{nik}^{(q)}(r', \theta', z') r'dr'd\theta'dz' \qquad (59)$$

Table 15 The Neutron Flux Distribution

Subsystem (q)	(n i k)	$\xi^{(q)}_{nik}$	$d^{(q)}_{nik}$	$a^{(q)}_{nik}$
1	1 0 1 1 2 1	0.622012×10^{5} -0.887950×10^{4}	3.8955 -5.9530×10^{-1}	1.2706×10^{4} -2.1941×10^{4}
2	1 2 1 1 4 1	-0.513175×10^{4} -0.256466×10^{4}	-3.6018×10^{-1} -1.8795×10^{-1}	-8.0206×10^{3} -3.1583×10^{3}
3	1 2 1 1 4 1	-0.887950×10^{4} -0.325026×10^{4}	-5.9530×10^{-1} -2.2745×10^{-1}	-2.1941×10^{4} -4.2072×10^{3}
4	1 0 1 1 2 1	-0.151127×10^{5} -0.513175×10^{4}	-9.8827×10^{-1} -3.6018×10^{-1}	-1.2888×10^{6} -8.0206×10^{3}
5	1 1 1 1 3 1	-0.821102×10^{4} -0.351564×10^{4}	-5.5932×10^{-1} -2.5267×10^{-1}	-1.8633×10^{4} -4.7043×10^{3}
6	1 1 1 1 3 1	-0.252902×10^{5} -0.494566×10^{4}	-1.6469 -3.3942×10^{-1}	3.9097×10^{4} -7.4869×10^{3}
7	1 1 1 1 3 1	-0.821102×10^{4} -0.351564×10^{4}	-5.5932×10^{-1} -2.5267×10^{-1}	-1.8633×10^{4} -4.7043×10^{3}
8	1 1 1 1 3 1	-0.252902×10^{5} -0.494566×10^{4}	-1.6469 -3.3942×10^{-1}	3.9097×10^{4} -7.4869×10^{3}

Table 16 The Expansion Coefficients $a^{(q)}_{nik}$

5. The Modal Expansion

From the discussion of the previous section it is clear that the neutron flux ψ_ϕ can be expressed in terms of the function f in the operator form;

$$\psi_\phi = Ff \tag{60}$$

where F is the operator (or an approximation to) $[I + \hat{L}]^{-1}L$. In the particular case of the example, the equation (60) corresponds to the equation (56).

Substituting equation (3) in (60) one finds

$$\psi_\phi = \frac{\sigma}{D} F\phi_o \psi_x + F\hat{u} \tag{61}$$

and eliminating the flux from equations (5) and (6) the dynamic equations for xenon and iodine are obtained as follows,

$$\frac{\partial \psi_I}{\partial t} = -\lambda_I \psi_I + \gamma_I \Sigma_f \frac{\sigma}{D} F\phi_o \psi_x + \gamma_I \Sigma_f F\hat{u} \tag{62}$$

$$\frac{\partial \psi_x}{\partial t} = \lambda_I \psi_I - (\lambda_x + \sigma\phi_o + \frac{\sigma^2}{D} x_o F\phi_o)\psi_x - \sigma x_o F\hat{u} \tag{63}$$

with the initial conditions (7) and (8).

Equations (62) and (63) can be written as an operator equation in the product space $L_2[V] \times L_2[V]$;

$$\frac{\partial}{\partial t} \underline{\psi} = \underline{\underline{A}}\,\underline{\psi} + \underline{z} \tag{64}$$

where

$$\underline{\psi} = \mathrm{COL}[\psi_I, \psi_x] \tag{65}$$

$$\underline{z} = \mathrm{COL}[z_I, z_x] \tag{66}$$

$$z_I = \gamma_I \Sigma_f F \hat{u} \tag{67}$$

$$z_x = - x_o \sigma F \hat{u} \tag{68}$$

and

$$\underline{A} = \begin{bmatrix} -\lambda_I & (\gamma_I \Sigma_f \frac{\sigma}{D} F\phi_o) \\ \lambda_I & -(\lambda_x + \sigma\phi_o + x_o \frac{\sigma^2}{D} F\phi_o) \end{bmatrix} \tag{69}$$

At this stage it is convenient to pause and make the following remark: note that the linear analysis of xenon spatial oscillation could be carried out by computing the eigenvalues of the operator $\underline{A}$. In fact this procedure would be equivalent to studying the xenon stability by means of Kaplan's natural modes [7]. Several difficult problems, however, of theoretical and computational nature make this approach impractical. A simple computation would show that the operator $\underline{A}$ is not compact, non-self adjoint and furthermore, not even normal. The theoretical implication of this, is that nothing can be said about the existence and completeness of the eigenvalues of the operator $\underline{A}$. Even if it is assumed that the eigenvalue spectrum of $\underline{A}$ is discrete, the numerical computation of the eigenfunctions and their corresponding eigenvalues is difficult. Successful computations have been reported only in the case of a slab reactor [11].

Because of this a number of methods have been proposed to obtain approximations to the natural modes. Of these, probably the best known are the Lamda and Mu methods developed by Stacey [10,15,30,31]. Roughly speaking, these methods consist of Laplace transforming equation (64) and manipulating the resulting equations in a way that allows the use of conventional, static diffusion codes.

The objective in this section is not to perform a stability analysis but to find an approximate solution to equation (64) in terms of the control variable $\hat{u}$. It is obvious however that the two problems are interrelated.

The method proposed consists of expanding the iodine and xenon concentrations in terms of a finite number of the Laplacian modes $\{\psi_j\}$ and obtaining a weak solution to equations (63) and (64) in the form of an integral equation. Let ΩI denote the finite set of subscripts corresponding to the modes in the finite expansion and let $\tilde{\psi}_I$ and $\tilde{\psi}_x$ denote the approximate solutions,

$$\tilde{\psi}_I = \sum_{i\varepsilon\Omega I} \psi_i \, W_{I_i} \tag{70}$$

$$\tilde{\psi}_x = \sum_{i\varepsilon\Omega I} \psi_i \, W_{x_i} \tag{71}$$

where the coefficients W_{I_i} and W_{x_i} satisfy the system of ordinary differential equations;

$$\frac{d}{dt} W_{I_i}(t) = -\lambda_I W_{I_i}(t) + \sum_{j\varepsilon\Omega I} \langle \psi_i, \gamma_I \Sigma_f \frac{\sigma}{D} F \phi_o \psi_j \rangle W_{x_j}(t) + \langle \psi_i, z_I(t) \rangle \tag{72}$$

and

$$\frac{d}{dt} W_{x_i}(t) = \lambda_I W_{I_i}(t) - \sum_{j\epsilon\Omega I} \langle \psi_j, (\lambda_x + \sigma\phi_o + \frac{\sigma^2}{D} x_o F\phi_o)\psi_j \rangle W_{x_j}(t)$$

$$+ \langle \psi_j, z_x(t) \rangle \qquad (73)$$

with the intial conditions;

$$W_{I_i}(t_o) = \langle \psi_i, \psi_I(t_o) \rangle \qquad (74)$$

$$W_{x_i}(t_o) = \langle \psi_i, \psi_x(t_o) \rangle \qquad (75)$$

Multiplying equations (72) and (73) by ψ_i and summing over the set ΩI, it can be seen that $\tilde{\psi}_I$ and $\tilde{\psi}_x$ satisfy the equations;

$$\frac{\partial}{\partial t} \tilde{\psi}_I = -\lambda_I \tilde{\psi}_I + \sum_{i,j\epsilon\Omega I} \psi_i \langle \psi_i \gamma_I \Sigma_f \frac{\sigma}{D} F\phi_o \psi_j \rangle \langle \psi_j, \tilde{\psi}_x \rangle$$

$$+ \sum_{i\epsilon\Omega I} \psi_i \langle \psi_i, z_I \rangle \qquad (76)$$

$$\frac{\partial}{\partial t} \tilde{\psi}_x = \lambda_I \tilde{\psi}_x - \sum_{i,j\epsilon\Omega I} \psi_i \langle \psi_i, (\lambda_x + \sigma\phi_o + x_o \frac{\sigma^2}{D} F\phi_o)\psi_j \rangle \langle \psi_j, \tilde{\psi}_x \rangle$$

$$+ \sum_{i\epsilon\Omega I} \psi_i \langle \psi_i, z_x \rangle \qquad (77)$$

In order to evaluate an estimate of the approximating error it is convenient to combine (76) and (77) in an operator equation similar to (64). To this end the following definitions are introduced;

$$\tilde{z}_1 = \sum_{i\epsilon\Omega I} \psi_i \langle \psi_i, z_I \rangle \tag{78}$$

$$\tilde{z}_2 = \sum_{i\epsilon\Omega I} \psi_i \langle \psi_i, z_x \rangle \tag{79}$$

Also, the operators $\tilde{A}_1$ and $\tilde{A}_2$ with range and domain in $L_2[V]$ are defined by

$$\tilde{A}_1\tilde{\psi}_x = \sum_{i,j\epsilon\Omega I} \psi_i \langle \psi_i, \gamma_I \Sigma_f + \frac{\sigma}{D} F\phi_o\psi_j \rangle \langle \psi_j, \tilde{\psi}_x \rangle \tag{80}$$

and

$$\tilde{A}_2\tilde{\psi}_x = \sum_{i,j\epsilon\Omega I} \psi_i \langle \psi_i, (\lambda_x + \sigma\phi_o + \frac{\sigma^2}{D} X_o F\phi_o)\psi_j \rangle \langle \psi_j, \tilde{\psi}_x \rangle \tag{81}$$

Finally, let

$$\underline{\tilde{\psi}} = \mathrm{COL}[\tilde{\psi}_I, \tilde{\psi}_x] \tag{82}$$

$$\underline{\tilde{z}} = \mathrm{COL}[\tilde{z}_I, \tilde{z}_x] \tag{83}$$

With these definitions, equations (76) and (77) are rewritten in

the form;

$$\frac{\partial}{\partial t}\ \tilde{\underline{\psi}} = \tilde{\underline{A}}\ \underline{\psi} + \tilde{\underline{z}} \tag{84}$$

where

$$\tilde{\underline{A}} = \begin{bmatrix} \lambda_I & \tilde{A}_1 \\ -\lambda_I & \tilde{A}_2 \end{bmatrix} \tag{85}$$

In view of equations (64) and (84), it follows that the error $\underline{\varepsilon} = \underline{\psi} - \tilde{\underline{\psi}}$ between the true solution to equation (64) and $\tilde{\underline{\psi}}$ satisfies

$$\frac{\partial}{\partial t}\ \underline{\varepsilon} = \underline{A}\ \underline{\psi} - \tilde{\underline{A}}\ \tilde{\underline{\psi}} + \underline{z} - \tilde{\underline{z}}$$

$$= \underline{A}\ \underline{\varepsilon} + (A-\tilde{A})\ \tilde{\underline{\psi}} + (z-\tilde{z}) \tag{86}$$

with initial condition

$$\underline{\varepsilon}(t_o) = (\underline{\psi}(t_o) - \tilde{\underline{\psi}}(t_o)) \tag{87}$$

If $\underline{A}$ is expressed as the addition of two matrix operators in $L_2[V] \times L_2[V]$,

$$\underline{A} = \underline{A}_o + \hat{\underline{A}} \tag{88}$$

where $\underline{A}_o$ is a two-dimensional matrix with constant coefficients then

the equation (86) could be written in the form

$$\frac{\partial}{\partial t}\underline{\varepsilon} = A_o\,\underline{\varepsilon} + \hat{A}\,\underline{\varepsilon} + \underline{h} \tag{89}$$

where

$$\underline{h} = (\underline{A}-\tilde{\underline{A}})\,\tilde{\underline{\psi}} + (\underline{z}-\tilde{\underline{z}}) \tag{90}$$

Then using the familiar convolution integral,

$$\underline{\varepsilon}(t) = \underline{g}(t) + \int_o^t \underline{\phi}_o(t,\tau)\,\hat{\underline{A}}\,\underline{\varepsilon}(\tau)d\tau \tag{91}$$

where $\underline{\phi}_o(t,\tau)$ is the fundamental matrix associated with $\underline{A}_o$ and where

$$\underline{g}(t) = \underline{\phi}_o(t,t_o)\,\underline{\varepsilon}(t_o) + \int_{t_o}^t \underline{\phi}_o(t,\tau)\,\underline{h}(\tau)d\tau \tag{92}$$

Equation (91) has the familiar form of the nonhomogeneous Volterra's equation of the second kind in the product space $L_2[V] \times L_2[V]$. It is well known that a unique solution to (91) exists provided that the operator $\underline{\phi}_o(t,\tau)\hat{\underline{A}}$ is bounded in $L_2[V] \times L_2[V]$ for all finite values of $(t-\tau)$. Furthermore, the solution can be expressed in the Neumann series (contraction mapping series);

$$\underline{\varepsilon}(t) = \underline{g}(t) + \sum_{n=1}^{\infty} \int_{t_o}^t \underline{K}_n(t,\tau)\,\underline{g}(\tau)d\tau \tag{93}$$

where

$$\underline{K}_1(t,\tau) = \underline{\phi}_o(t,\tau)\hat{\underline{A}} \tag{94}$$

and

$$\underline{K}_n(t,\tau) = \int_\tau^t \underline{\phi}_o(t, s)\, \hat{\underline{A}}\, \underline{K}_{n-1}(s,\tau)ds \tag{95}$$

Consider a real, bounded and positive number M such that for all $(t-\tau)$;

$$||\underline{K}_1(t,\tau)\ \underline{g}(\tau)||_2 \leq M\ ||\underline{g}(\tau)||_2 \tag{96}$$

where $||\cdot||_2$ denotes the norm in $L_2[V] \times L_2[V]$ inherited from the inner product $<\ ,\ >_2$;

$$<\underline{g},\ \underline{g}>_2 = <g_1,\ g_1> + <g_2,\ g_2> \tag{97}$$

and let

$$M_g = \underset{t}{\text{Max}}||\underline{g}(t)||_2 \tag{98}$$

then it follows that

$$||\underline{K}_2(t,\ \tau)\ \underline{g}(\tau)||_2 \leq \int_\tau^t ||K_1(t,\ s)||_2\ M\ M_g\ ds$$

$$= (t-\tau)\ M^2 M_g \tag{99}$$

and by induction;

$$||\underline{K}_n(t, \tau)\, \underline{g}(\tau)||_2 \leq \frac{(t-\tau)^{n-1}}{(n-1)!}\, M^n\, M_g \qquad (100)$$

which tends to zero as n tends to infinity. Using (100) it follows from (93) that

$$||\underline{\varepsilon}(t)||_2 \leq ||\underline{g}(t)||_2 + \sum_{n=1}^{\infty} \int_{t_o}^{t} ||\underline{K}_n(t, \tau)\underline{g}(\tau)||d\tau$$

$$\leq [1 + \sum_{n=1}^{\infty} \frac{(t-to)^n}{n!}\, M^n]\, M_g$$

$$= e^{(t-t_o)\, M}\, M_g \qquad (101)$$

Even though this error estimate is very conservative, together with (90) and (92) it can be of help in determining how good the approximation $\tilde{\underline{\psi}}$ is. In particular, (90) is a good criterion for checking the adequacy of the approximation. Note also that the sharpest estimate obtainable by this approach would be determined by choosing $\underline{A}_o$ so as to minimize the bound M in (96).

5.1 The Functional Relation

Using the familiar convolution integral, the solution to the system of equations (72) and (73) is obtained in the form;

$$W_{I_i}(t) = \sum_{j\epsilon\Omega I} (\phi_{II_{ij}}(t, t_o)\, W_{I_j}(t_o) + \phi_{IX_{ij}}(t, t_o)\, W_{x_j}(t_o))$$

$$+ \sum_{j\epsilon\Omega I} \int_{t_o}^{t} (\phi_{II_{ij}}(t, \tau) \langle \psi_j, z_I(\tau)\rangle + \phi_{IX_{ij}}(t, \tau)\langle \psi_j, z_x(\tau)\rangle)d\tau \tag{102}$$

and

$$W_{x_i}(t) = \sum_{j\epsilon\Omega I} (\phi_{XI_{ij}}(t, t_o)\, W_{I_j}(t_o) + \phi_{xx_{ij}}(t, t_o)\, W_{x_j}(t_o))$$

$$+ \sum_{j\epsilon\Omega I} \int_{t_o}^{t} (\phi_{XI_{ij}}(t, \tau) \langle \psi_j, z_I(\tau) + \phi_{xx_{ij}}(t, \tau)\langle \psi_j, z_x(\tau)\rangle)d\tau \tag{103}$$

where the time-dependent functions $\phi_{II_{ij}}$, $\phi_{IX_{ij}}$, $\phi_{XI_{ij}}$, and $\phi_{XX_{ij}}$ are the components of the matrices Φ_{II}, Φ_{IX}, Φ_{XI} and Φ_{XX} which form the fundamental matrix

$$\Phi = \begin{bmatrix} \Phi_{II} & \Phi_{IX} \\ \Phi_{XI} & \Phi_{XX} \end{bmatrix} \tag{104}$$

associated with

$$\begin{bmatrix} A_{II} & A_{IX} \\ A_{XI} & A_{XX} \end{bmatrix} \tag{105}$$

where A_{II}, A_{IX}, A_{XI} and A_{XX} are square matrices of dimension equal to the number of subscripts in ΩI and where

$$A_{II} = \text{diag}\,[-\lambda_I] \tag{106}$$

$$A_{XI} = -A_{II} \tag{107}$$

and finally A_{IX} and A_{XX} have components $a_{IX_{ij}}$ and $a_{XX_{ij}}$ defined by

$$a_{IX_{ij}} = \langle \psi_i,\ \gamma_I \Sigma_f \frac{\sigma}{D} F\, \phi_o\, \psi_j \rangle \tag{108}$$

and

$$a_{XX_{ij}} = -\langle \psi_i,\ (\lambda_x + \sigma\phi_o + \frac{\sigma^2}{D} X_o\, F\phi_o)\psi_j \rangle \tag{109}$$

Substituting (67) and (68) in (103) the coefficients of the xenon expansion are obtained in terms of the control variable $\hat{u}$;

$$W_{x_i}(t) = \sum_{j\epsilon\Omega I} (\phi_{XI_{ij}}(t, t_o)\langle \psi_j, \psi_I(t_o)\rangle$$

$$+ \phi_{XX_{ij}}(t, t_o)\langle \psi_j, \psi_x(t_o)\rangle) + \sum_{j\epsilon\Omega I} \int_{to}^{t} (\phi_{XI_{ij}}(t, \tau)\langle \psi_j, \gamma_I \Sigma_f F\hat{u}(\tau)\rangle$$

$$- \phi_{XX_{ij}}(t, \tau)\langle \psi_j, X_o \sigma F\, \hat{u}(\tau)\rangle)d\tau \tag{110}$$

recalling from (61) that

$$\psi_\phi = \frac{\sigma}{D} F \phi_o \psi_x + F \hat{u} \tag{61}$$

and since $\psi_x = \sum_{i\epsilon\Omega I} \psi_i W_{x_i}$, the neutron flux is obtained in terms of the control variable $\hat{u}$;

$$\psi_\phi(t) = \sum_{i\epsilon\Omega I} \frac{\sigma}{D} F \phi_o \psi_i W_{x_i}(t) + F \hat{u}(t) \tag{111}$$

where for simplicity only the temporal dependency is explicitly shown.

Substituting (110) in (111), the desired functional relation is obtained;

$$\psi_\phi = \Delta + F \hat{u} + F_1 \hat{u} \tag{112}$$

where Δ is a function in $L_2[V]$ defined by

$$\Delta(t) = \sum_{i\epsilon\Omega I} [\frac{\sigma}{D} F\phi_o\psi_i][\sum_{j\epsilon\Omega I} (\phi_{XI_{ij}}(t, t_o)<\psi_j, \psi_I(t_o)>$$

$$+ \phi_{XX_{ij}}(t, t_o) <\psi_j, \psi_x(t_o)>] \tag{113}$$

and F_1 is the transformation from the space of control functions into $L_2[V]$;

$$F_1 \hat{u}(t) = \sum_{i,j\epsilon\Omega I} [\frac{\sigma}{D} F\phi_o\psi_i][\int_{t_o}^{t} (\phi_{XI_{ij}}(t, \tau)<\psi_j, \nu_I\Sigma_f F \hat{u}(\tau)>$$

$$- \phi_{XX_{ij}}(t, \tau)<\psi_j, X_o\sigma F \hat{u}(\tau)>)d\tau \tag{114}$$

5.2 An Example

For the particular case of the example in section 3.2 the matrices $A_{II}^{(q)}$, $A_{IX}^{(q)}$, $A_{XI}^{(q)}$ and $A_{XX}^{(q)}$ corresponding to the q-th subsystem are diagonal and two-dimensional. The diagonal entries in $A_{IX}^{(q)}$ and $A_{XX}^{(q)}$;

$$a_{IX_{(nik)}}^{(q)} = \langle \psi_{nik}^{(q)} , \gamma_I \Sigma_f \frac{\sigma}{D} F \phi_o \psi_{nik}^{(q)} \rangle \qquad (115)$$

and

$$a_{XX_{(nik)}}^{(q)} = - \langle \psi_{nik}^{(q)} , (\lambda_x + \sigma\phi_o + \frac{\sigma^2}{D} X_o F\phi_o) \psi_{nik}^{(q)} \rangle \qquad (116)$$

are given in Table 17, in which the eigenvalues $p_{1_{(nik)}}^{(q)}$ and $p_{2_{(nik)}}^{(q)}$ of the augmented matrix

$$\begin{bmatrix} A_{II}^{(q)} & A_{IX}^{(q)} \\ A_{XI}^{(q)} & A_{XX}^{(q)} \end{bmatrix} \qquad (117)$$

are also shown.

It can be seen from the eigenvalues of Table 17 that although all the subsystems exhibit oscillatory behavior, all but one of the subsystems are stable. The mode $\psi_{101}^{(4)}$ is unstable. Of all the stable modes, $\psi_{111}^{(6)}$ exhibits the slowest transient response with a time constant of 47.13 hours.

The neutron flux ψ_ϕ is obtained in the form

$$\psi_\phi = \sum_{q=1}^{8} \psi_\phi^{(q)} \qquad (118)$$

Subsystem q	nik	$a_{IX(nik)}^{(q)}$	$a_{XX(nik)}^{(q)}$	Eigenvalues $p_{1(nik)}^{(q)}$	$p_{2(nik)}^{(q)}$
1	101	6.2988×10^{-5}	-1.5289×10^{-4}	-1.5538×10^{-5}	-1.6605×10^{-5}
	121	-7.5823×10^{-5}	-2.5597×10^{-5}	$(-2.7149+j4.6623)\times10^{-5}$	$(-2.7149-J4.6623)\times10^{-5}$
2	121	-2.2174×10^{-5}	-5.2846×10^{-5}	$(-4.0773+j2.2150)\times10^{-5}$	$(-4.0773-j2.2150)\times10^{-5}$
	141	-6.9615×10^{-5}	-5.3694×10^{-5}	$(-4.1197+j4.2916)\times10^{-5}$	$(-4.1197-j4.2916)\times10^{-5}$
3	121	-7.5823×10^{-5}	-2.5597×10^{-5}	$(-2.7149+j4.6623)\times10^{-5}$	$(-2.7149-j4.6623)\times10^{-5}$
	141	-1.1593×10^{-5}	-5.9523×10^{-5}	$(-4.4112+j0.9757)\times10^{-5}$	$(-4.4112+j0.9757)\times10^{-5}$
4	101	-5.1112×10^{-3}	3.6426×10^{-3}	3.6022×10^{-3}	1.1701×10^{-5}
	121	-2.2174×10^{-5}	-5.2846×10^{-5}	$(-4.0773+j2.2150)\times10^{-5}$	$(-4.0773-j2.2150)\times10^{-5}$
5	111	-6.0093×10^{-5}	-3.3855×10^{-5}	$(-3.1278+j4.1449)\times10^{-5}$	$(-3.1278-j4.14449)\times10^{-5}$
	131	-1.1497×10^{-5}	-5.4727×10^{-5}	$(-4.1714+j1.2673)\times10^{-5}$	$(-4.1714-j1.2673)\times10^{-5}$
6	111	1.5761×10^{-4}	-2.0423×10^{-4}	-5.8933×10^{-6}	-2.2704×10^{-4}
	131	-2.2872×10^{-5}	-5.6661×10^{-5}	$(-4.2681+j2.1470)\times10^{-5}$	$(-4.2681-j2.1470)\times10^{-5}$
7	111	-6.0093×10^{-5}	-3.3855×10^{-5}	$(-3.1278+j4.1449)\times10^{-5}$	$(-3.1278-j4.1449)\times10^{-5}$
	131	-1.1497×10^{-5}	-5.4727×10^{-5}	$(-4.1714+j1.2673)\times10^{-5}$	$(-4.1714-j1.2673)\times10^{-5}$
8	111	1.5761×10^{-4}	-2.0423×10^{-4}	-5.8933×10^{-6}	-2.2704×10^{-4}
	131	-2.2872×10^{-5}	-5.6661×10^{-5}	$(-4.2681+j2.1470)\times10^{-5}$	$(-4.2681-j2.1470)\times10^{-5}$

Table 17 System Eigenvalues

where

$$\psi_{\phi}^{(q)} = \Delta^{(q)} + F^{(q)}\hat{u}^{(q)} + F_1^{(q)}\hat{u}^{(q)} \tag{119}$$

Expanding the control variable $\hat{u}^{(q)}$ in terms of the symmetry-adapted linear combinations $\{\rho_k^{(q)}\}$ k=1, C_q of Chapter V;

$$\hat{u}^{(q)}(t) = \sum_{k=1}^{C_q} \phi_o \rho_k^{(q)} y_k^{(q)}(t) \tag{120}$$

and using the spectral representation of $F^{(q)}$ as given in Table 15 the terms $\Delta^{(q)}$, $F^{(q)}\hat{u}^{(q)}$ and $F_1^{(q)}\hat{u}^{(q)}$ are given as follows;

$$\Delta^{(q)}(t) = \sum_{(nik)\in\Omega(q)} \frac{\sigma}{D} \psi_{nik}^{(q)} A_{nik}^{(q)} \langle \psi_{nik}^{(q)}, \phi_o \psi_{nik}^{(q)} \rangle \Delta_{nik}^{(q)}(t) \tag{121}$$

where

$$\Delta_{nik}^{(q)}(t) = \phi_{XI_{nik}}^{(q)}(t, t_o)\langle \psi_{nik}^{(q)}, \psi_I(t_o)\rangle$$

$$+ \phi_{XX_{nik}}^{(q)}(t, t_o)\langle \psi_{nik}^{(q)}, \psi_X(t_o)\rangle \tag{122}$$

and the time-dependent functions ϕ_{XI} and ϕ_{XX} are defined by

$$\phi_{XI_{nik}}^{(q)}(t-\tau) = \frac{-\lambda_I}{(p_{1_{nik}}^{(q)} - p_{2_{nik}}^{(q)})} \left[e^{p_{1_{nik}}^{(q)}(t-\tau)} - e^{p_{2_{nik}}^{(q)}(t-\tau)} \right] \tag{123}$$

and

$$\phi^{(q)}_{XX_{nik}}(t-\tau) = \frac{1}{(p_{1_{nik}}^{(q)} - p_{2_{nik}}^{(q)})}\left[(p_{1_{nik}}^{(q)} + \lambda_I)\, e^{p_{1_{nik}}^{(q)}(t-\tau)}\right.$$

$$\left. - (p_{2_{nik}}^{(q)} + \lambda_I)\, e^{p_{2_{nik}}^{(q)}(t-\tau)}\right] \qquad (124)$$

if the eigenvalues are real or;

$$\phi^{(q)}_{XI_{nik}}(t-\tau) = \frac{-\lambda_I}{IM[p_{1_{nik}}^{(q)}]}\, e^{RE[p_{1_{nik}}^{(q)}](t-\tau)}\, \sin(IM[p_{1_{nik}}^{(q)}](t-\tau)) \qquad (125)$$

and

$$\phi^{(q)}_{XX_{nik}}(t-\tau) = h^{(q)}_{nik}\, e^{RE[p_{1_{nik}}^{(q)}](t-\tau)}\, \sin(IM[p_{1_{nik}}^{(q)}](t-\tau) + \phi^{q}_{nik}) \qquad (126)$$

where

$$h^{(q)}_{nik} = \frac{((RE[p_{1_{nik}}^{(q)}] + \lambda_I)^2 + IM[p_{1_{nik}}^{(q)}]^2)^{1/2}}{IM[p_{1_{nik}}^{(q)}]} \qquad (127)$$

and

$$\phi^{(q)}_{nik} = \tan^{-1}\left(\frac{IM[p_{1_{nik}}^{(q)}]}{RE[p_{1_{nik}}^{(q)} + \lambda_I]}\right) \qquad (128)$$

also

$$F^{(q)}\hat{u}^{(q)} = \sum_{m=1}^{Cq} \sum_{(nik)\epsilon\Omega^q} \psi_{nik}^{(q)} a_{nik}^{(q)} \langle \psi_{nik}^{(q)}, \phi_o \rho_m^{(q)} \rangle y_m^{(q)}(t) \tag{129}$$

where the coefficients are given in Tables 17, 18 and 19.

Finally, $F_1^{(q)} \hat{u}^{(q)}$ is obtained as,

$$F_1^{(q)}\hat{u}^{(q)} = \sum_{(nik)\epsilon\Omega^{(q)}} \psi_{nik}^{(q)} a_{nik}^{(q)} \frac{\sigma}{D} \langle \psi_{nik}^{(q)}, \phi_o \psi_{nik}^{(q)} \rangle f_{1_{nik}}^{(q)}(t) \tag{130}$$

and

$$f_{1_{nik}}^{(q)}(t) = \sum_{m=1}^{Cq} \int_{t_o}^{t} (\phi_{XI_{nik}}^{(q)}(t-\tau) \langle \psi_{nik}^{(q)}, \phi_o \rho_m^{(q)} \rangle a_{nik}^{(q)} \gamma_I \Sigma_f$$

$$- \phi_{XX_{nik}}^{(q)}(t-\tau) \langle \psi_{nik}^{(q)}, X_o \psi_{nik}^{(q)} \rangle a_{nik}^{(q)} \langle \psi_{nik}^{(q)}, \phi_o \rho_m^{(q)} \rangle \sigma) y_m^{(q)}(\tau) d\tau \tag{131}$$

where the coefficients are given in Tables 17, 18 and 19.

Subsystem q	nik	$\langle\psi^{(q)}_{nik}, \phi_o\psi^{(q)}_{nik}\rangle \times 10^{-14}$	$\langle\psi^{(q)}_{nik}, X_o\psi^{(q)}_{nik}\rangle \times 10^{-14}$
1	101	0.6812	0.4686
	121	0.4784	0.4239
2	121	0.3799	0.3921
	141	0.3029	0.3607
3	121	0.4784	0.4239
	141	0.3786	0.3942
4	101	0.5449	0.4406
	121	0.3799	0.3921
5	111	0.4431	0.4131
	131	0.3358	0.3751
6	111	0.5539	0.4434
	131	0.4198	0.4079
7	111	0.4431	0.4131
	131	0.3358	0.3751
8	111	0.5539	0.4434
	131	0.4198	0.4079

Table 18 Inner Product Values

Subsystem q	nik	$\langle \psi^{(q)}_{nik}, \phi_o \rho^{(q)}_j \rangle$ j=1	j=2	j=3
1	101 121	0.6331×10^{11} 0.3845×10^{11}	0.1954×10^{14} -0.3839×10^{14}	0.8030×10^{14} -0.4175×10^{13}
2	121 141	0.1686×10^{12} 0.8859×10^{11}		
3	121 141	0.1192×10^{12} 0.6264×10^{11}		
4	101 121	0.8953×10^{11} 0.5437×10^{11}	0.2764×10^{14} -0.5430×10^{14}	0.1136×10^{15} -0.5904×10^{13}
5	111 131	0.1336×10^{12} -0.5708×10^{11}		
6	111 131	0.6805×10^{11} 0.1164×10^{12}	0.3554×10^{14} -0.3853×10^{14}	
7	111 131	0.9624×10^{11} 0.1646×10^{12}	0.5026×10^{14} -0.5448×10^{14}	
8	111 131	0.9448×10^{11} -0.4036×10^{11}		

Table 19 Inner Product Values

Number of Rods			
Rod Diameter			
Rod Length			
Location of Rod Centers			
Rod 1	z = 150 cm	θ = 0.6747 Rad	r = 255 cm
Rod 2	z = 150 cm	θ = π/2 Rad	r = 255 cm
Rod 3	z = 150 cm	θ = π/2 Rad	r = 0 cm
Rods 4 to 14 are located symmetrically as shown in Figure 7.			

Table 20 Control Rod Data

CHAPTER VII

CONCLUDING REMARKS

1 Summary

The optimization technique of the minimum norm formulation in Hilbert spaces has been applied to a variety of control problems in which a linear distributed parameter model is utilised to describe the dynamic behavior of a nuclear reactor core.

All the control problems considered here are variations on the same theme; the control of the reactor's state distribution in the neighborhood of a given state.

To the best of the author's knowledge, only the solution to the problem of minimizing a performance index that penalizes both the deviations of the reactor's state and the control effort has been published in the literature by other authors.

While the more conventional techniques approach this problem through modal expansion methods and invoke variational principles which yield necessary conditions for optimality in the form of an infinite system of ordinary differential equations with mixed boundary conditions, the minimum norm formulation employed here yields necessary and sufficient conditions for optimality in the form of a finite set of coupled integral equations which can be solved by means of iterative techniques.

It has been shown in Chapter IV how the additional constraints, imposed on both the total power generated in the core and the terminal state, can be treated by the optimization technique of the minimum norm. Also, it has been demonstrated how these additional constraints

can be related so that the resulting problems are suboptimal.

The application of the minimum norm approach to all the problems considered in the thesis was described in terms of a general linear distributed parameter model. Several examples were given in which the particular case of a homogeneous slab reactor with a finite number of controls and with the one-energy neutronic model is considered.

It was demonstrated in Chapter V how by merely studying the geometrical symmetry of a nuclear reactor core it is possible to reduce the mathematical model of the core into more tractable submodels of lower order. This simplification of the problem will facilitate the practical implementation of the minimum norm techniques to more realistic reactor models which may involve, for example, geometrical configurations in two or three dimensions.

A new method was developed in Chapter VI for obtaining an approximate solution to a reactor core model that is suitable for describing the dynamic behavior of the reactor core during load following operations.

The method, which can be classified as a modal expansion approach using the eigenfunctions of the Laplacian operator, yields the solution to the model in an operator form that is amenable to application of the minimum norm technique.

The proposed method has a scope of application that covers those cases where the reactor core is of simple geometric configuration, so that the eigenfunctions of the Laplacian operator are explicitly known, and where the diffusion coefficients are nearly homogeneous.

The method was described in terms of the modified one-energy

neutronic model with xenon and iodine dynamics.

Both, the symmetry reduction procedure and the modal expansion approach were combined in an example that involves a cylindrical reactor and utilizes the numerical data typical of a large nuclear power reactor of the thermal type.

2 Suggestions for Further Work

The application of the minimum norm technique to situations in which the saturation of the control functions is also taken into account constitutes a natural extension to the present work. While in theory this type of problem can be formulated and treated in the context of functional analysis, the implementation of the optimizing algorithms is difficult. More work is needed in this area.

An ideal approach to the optimal control of linear distributed parameter systems should combine the computational advantages of the methods of functional analysis with the engineering benefits of the feedback control concept, which characterizes the Kalman-Pontryagin theory of optimal control for lumped linear systems.

It would be worthwhile to investigate, in this spirit, the feasibility of synthesizing practical feedback control schemes for distributed parameter systems via functional analysis.

Also needed is a sensitivity analysis of the optimal control algorithms reported here with respect to model inaccuracies and system's noise.

Further work is needed to evaluate trade-offs in distributed model sophistication versus worth and practical implementability of the optimizing algorithms.

A different variety of problems, that is also of interest to the electric power industry, to which the application of the minimum norm is worth investigating arises in the area of fuel consumption optimization in continuously fueled nuclear reactors [98, 99, 100, 101].

The basic problem in this area consists of finding the optimum power distribution, and the corresponding neutron flux, throughout the reactor core so that the fuel burn up is maximized.

Although variational calculus has been applied to some particular cases in this area, it may be possible that the minimum norm technique leads to more accurate and efficient algorithms.

The short-range economic operation optimization of electric power systems with nuclear, hydro and thermal generating plants is another promising area for application of the minimum norm technique.

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APPENDIX I

Nuclear Data for a Typical 1200 MW(th) Natural Uranium, Heavy Water-Moderated, Pressurized-Tube Nuclear Reactor.

Infinite Multiplication Factor	k_∞	= 1.404027	
Resonance Scape Probability	p	= 0.89801	
Neutron Slowing Down Length	L_s^2	= 134.3	cm^2
Thermal Diffusion Length	L^2	= 235.4	cm^2
Fast Neutron Group Diffusion Coeff.	D_F	= 1.3643	cm
Slow Neutron Group Diffusion Coeff.	D_{th}	= 1.2349	cm
Fast Neutron Speed	V_f	= 10^7	cm/sec
Slow Neutron Speed	V	= $3x10^5$	cm/sec
Neutrons Produced Per Fission	v	= 2.640	
Core Radius	R	= 400	cm
Core Length	L	= 600	cm

Table 21 Two-Energy Neutronic Data

* Clean Reactor Data Provided to the Author in Private Communication by Dr. G.M. Frescura of the Ontario Hydro's Department of Nuclear Studies and Safety.

Fission Macroscopic Cross Section	$\Sigma_f = \frac{k_\infty \Sigma_{ath}}{pv}$
Absorption Macroscopic Cross Section	$\Sigma_{ath} = Dth/L^2$
Absorption Macroscopic Cross Section	$\Sigma_a = D_f/L_s^2$
Scattering Cross Section	$\Sigma_{1/2} = D_f(1-p)/L_s^2$

Table 22 Nuclear Cross Sections for the Two-Energy Neutronic Model

Delayed Neutron Precursor i	Relative Abundance β_i/β	Precursor Decay Constant λ_{c_i} (sec^{-1})
1	0.0405	0.000763
2	0.2129	0.0305
3	0.1855	0.1180
4	0.3882	0.3137
5	0.1268	1.226
6	0.0461	3.175

$$\beta = \sum_i \beta_i = 0.004982$$

Table 23 Delayed Neutron Precursor Data

Neutron Absorption Microscopic Cross Section of Xenon-135	$\sigma_x = 1.22 \times 10^{-18}$	cm^2
Xenon Decay Constant	$\lambda_x = 2.09 \times 10^{-5}$	sec^{-1}
Iodine Decay Constant	$\lambda_I = 2.87 \times 10^{-5}$	sec^{-1}
Iodine Yield Per Fission	$\gamma_I = 6.4 \times 10^{-2}$	

Table 24 Parameters for Dynamic Model of Xenon and Iodine

APPENDIX II

Derivation of the Norms of L, $\tilde{L}$, $(L-\tilde{L})$ and $\hat{L}$.

For convenience the Dyadic notation is used. That is, the symbol $\phi>$ denotes a function ϕ in $L_2[V]$. A functional ψ in $L_2[V]$ is represented in the form $<\psi$.

The spectral representation of L is given by

$$L = \sum_j \psi_j> \xi_j < \psi_j \tag{II.1}$$

where

$$\xi_j = \frac{1}{(b_o + \lambda_j)} \tag{II.2}$$

Also the Fourier representation of $\phi>$ is of the form;

$$\phi> = \sum_j \psi_j> <\psi_j, \phi> \tag{II.3}$$

$$= \sum_j \psi_j> C_j \tag{II.4}$$

where

$$C_j = <\psi_j, \phi> . \tag{II.5}$$

Given the orthonormality of the functions ψ_j, it follows that

$$L\phi> = \sum_j \psi_j> \xi_j C_j \tag{II.6}$$

and

$$||L\phi|| = [\sum_j (\xi_j^2 c_j^2)]^{1/2} \tag{II.7}$$

$$\leq \max_j |\xi_j| \; ||\phi|| \; . \tag{II.8}$$

Note also that the norm of L is defined by

$$||L|| = \sup_{||\phi||=1} ||\phi|| \; . \tag{II.9}$$

Consider a function ϕ with Fourier coefficients C_i, where;

$$C_i = \begin{cases} \dfrac{\max_j |\xi_j|}{\xi_j} & i = J_{max} \\ 0 & i \neq J_{max} \end{cases} \tag{II.10}$$

and J_{max} is the subscript corresponding to the term $\max_j |\xi_j|$, that is;

$$|\xi_{j_{max}}| = \max_j |\xi_j| \; . \tag{II.11}$$

It is clear that $||\phi||=1$, also

$$||L\phi|| = \max_j |\xi_j| \; . \tag{II.12}$$

In view of (II.9) and (II.12) it follows that

$$||L|| = \max_j |\xi_j| \tag{II.13}$$

similarly,

$$||\tilde{L}|| = \underset{j\in\Omega}{\text{Max}} \; |\xi_j| \,. \tag{II.14}$$

Consider now the operator $\hat{L} = \hat{L}b$. The norm of $\hat{L}\phi>$ is given by,

$$||\hat{L}\phi|| = <\hat{L}\phi, \hat{L}\phi>^{1/2}$$

$$= <\phi, \hat{L}^*\hat{L}\phi>^{1/2} \tag{II.15}$$

where the adjoint $\hat{L}^*$ is obtained in the form

$$\hat{L}^* = \hat{b}\, L \tag{II.16}$$

L is compact, $L\hat{b}$ and $\hat{L}^*\hat{L}$ are also compact. Since $\hat{L}^*\hat{L}$ is self adjoint it follows from the spectral theorem for compact, self adjoint operators that the spectrum of L*L consists of a finite or countably infinite set of eigenvalues and that the sequence of eigenfunctions of $\hat{L}^*\hat{L}$ is orthogonal and complete in $L_2[V]$, provided that zero is not in the spectrum of $\hat{L}^*\hat{L}$. Therefore, if $\{U_j\}$ and $\{\gamma_j\}$ denote respectively the sequence of eigenfunctions and the spectrum of $\hat{L}^*\hat{L}$, then each function ϕ in $L_2[V]$ has the Fourier representation

$$\phi> = \sum_j U_j> <U_j, \phi> \tag{II.17}$$

Also, the spectral representation of $\hat{L}^*\hat{L}$ is given by;

$$\hat{L}^*\hat{L} = \sum_j U_j> \gamma_j <U_j \tag{II.18}$$

therefore,

$$||\hat{L}\phi|| = \langle \phi, \hat{L}^*\hat{L}\phi \rangle^{1/2}$$

$$= [\sum_j \gamma_j \langle U_j, \phi \rangle^2]^{1/2}$$

$$\leq \max_j \gamma_j^{1/2} \, ||\phi|| \qquad \text{(II.19)}$$

and it follows that

$$||\hat{L}|| = \max_j \gamma_j^{1/2}. \qquad \text{(II.20)}$$